Errata

1 Chen & Kotlarchyk, ISBN: 981022026X, pp.112-113

Equation (4.99) does *not* vanish.

$$\int \mathbf{E}\cdot\nabla\phi d^3r \;=\; \int \nabla\cdot(\phi\mathbf{E})d^3r - \int \phi(\nabla\cdot\mathbf{E})d^3r$$

is still correct, and the first term on RHS indeed vanishes. However, the second term is,

$$-\int \phi(\nabla\cdot\mathbf{E})d^3r \;=\; -4\pi\int \phi\rho d^3r \;=\; -4\pi\sum_i e_i\phi_i,$$

and it is *non sequitur* to assume that the whole space is charge-free because this example *is* about charged particle-EM field coupling. In fact, we have,

$$\begin{aligned}
H &= H_{\text{particle}} + H_f \\
&= H_{\text{particle}} + \left\{\frac{1}{4\pi}\int \mathbf{E}\cdot\nabla\phi d^3r + \frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2)d^3r\right\} \\
&= \sum_i\left[\frac{\left|\mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i\right|^2}{2m_i} + e_i\phi_i\right] + \left\{\frac{1}{4\pi}\left(-4\pi\sum_i e_i\phi_i\right) + \frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2)d^3r\right\} \\
&= \sum_i \frac{1}{2m_i}\left|\mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i\right|^2 + \frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2)d^3r.
\end{aligned}$$

So equation (4.101) should be,

$$H \;=\; \frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{B}^2)d^3r \;+\; \sum_i \frac{1}{2m_i}\left|\mathbf{p}_i - \frac{e_i}{c}\mathbf{A}_i\right|^2.$$

What this means is that when a bunch of charged particles interact with each other via the EM field, the conserved quantity is their kinetic energies plus the integral of the EM field energy density (the energy of photons). There is no separate $\int\rho\phi d^3r$ term. In electrostatics, one could show that $\int\rho\phi d^3r/2$ is equal in value[1] to $\int\mathbf{E}^2d^3r/8\pi$, therefore adding $\int\rho\phi d^3r$ in H again would be double counting. This conservation law is also explicitly proved in equation (4.20).

[1] In fact, this equality is the reason to define $\mathbf{E}^2/8\pi$ as the electromagnetic field energy density.