2 Double Pendulum

Answer: assume $m_1=m_2=m$, $l_1=l_2=l$

(a) θ_1 , θ_2.

(b) The center of mass of the first rod is at

$$(x_1,y_1) = (\frac{l}{2}\sin\theta_1 \,,\ \frac{l}{2}\cos\theta_1)$$

The center of mass of the second rod is at

$$(x_2,y_2) = (l\sin\theta_1 + \frac{l}{2}\sin\theta_2 \,,\ -l\cos\theta_1 - \frac{l}{2}\cos\theta_2)$$

Therefore the total potential energy is,

$$V = mg\,(y_1 + y_2) = \frac{mgl}{2}(-3\cos\theta_1 - \cos\theta_2)$$

(c) From (b), we have,

$$(\dot{x}_1 + \dot{y}_1) = (\frac{l}{2}\cos\theta_1\dot{\theta}_1 \,,\ \frac{l}{2}\sin\theta_1\dot{\theta}_1)$$

$$(\dot{x}_2 + \dot{y}_2) = (l\cos\theta_1\dot{\theta}_1 + \frac{l}{2}\cos\theta_2\dot{\theta}_2 \,,\ l\sin\theta_1\dot{\theta}_1 + \frac{l}{2}\sin\theta_2\dot{\theta}_2)$$

Therefore, the kinetic energy of the center of mass is,

$$T_1 = \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{m}{2}(\frac{l^2}{4}\dot{\theta}_1^2) + \frac{m}{2}(l^2\dot{\theta}_1^2 + \frac{l^2}{4}\dot{\theta}_2^2 + l^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)$$

$$= \frac{ml^2}{2}(\frac{5}{4}\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2)$$

There are also rotational energies. The sum is,

$$T_2 = \frac{I_1}{2}\dot{\theta}_1^2 + \frac{I_2}{2}\dot{\theta}_2^2 = \frac{ml^2\dot{\theta}_1^2}{24} + \frac{ml^2\dot{\theta}_2^2}{24} = \frac{ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)}{24}$$

$$\left(I_1 = I_2 = 2\int_0^{l/2} r^2 m \frac{dr}{l} = \frac{ml^2}{12} \right)$$

Therefore, the total kinetic energy is,

$$T = T_1 + T_2 = \frac{ml^2}{2}\left(\frac{5}{4}\dot{\theta}_1^2 + \frac{1}{4}\dot{\theta}_2^2 + \cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2\right) + \frac{ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)}{24}$$

$$= \frac{2ml^2\dot{\theta}_1^2}{3} + \frac{ml^2\dot{\theta}_2^2}{6} + \frac{ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2}{2}$$

(d)

$$L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = \mathrm{T} - \mathrm{V}$$

$$= \frac{2ml^2\dot{\theta}_1^2}{3} + \frac{ml^2\dot{\theta}_2^2}{6} + \frac{ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2}{2} + \frac{mgl}{2}(3\cos\theta_1 + \cos\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{4ml^2\dot{\theta}_1}{3} + \frac{ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_2}{2}$$

$$\frac{\partial L}{\partial \theta_1} = -\frac{ml^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2}{2} - \frac{3mgl\sin\theta_1}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1} \quad \Rightarrow \ \ldots$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{ml^2\dot{\theta}_2}{3} + \frac{ml^2\cos(\theta_1 - \theta_2)\dot{\theta}_1}{2}$$

$$\frac{\partial L}{\partial \theta_2} = \frac{ml^2\sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2}{2} - \frac{mgl\sin\theta_2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2} \Rightarrow \ \ldots$$

(e) When θ_1 , $\theta_2 \sim 0$, keeping only the linear order terms,

$$\frac{\partial L}{\partial \dot{\theta}_1} = -\frac{3mgl\theta_1}{2}, \text{ and } \frac{\partial L}{\partial \dot{\theta}_1} = \frac{4ml^2\dot{\theta}_1}{3} + \frac{ml^2\dot{\theta}_2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \frac{\partial L}{\partial \theta_1} \Rightarrow \frac{4ml^2\ddot{\theta}_1}{3} + \frac{ml^2\ddot{\theta}_2}{2} = -\frac{3mgl\theta_1}{2}$$

$$\Rightarrow \frac{4l\ddot{\theta}_1}{3} + \frac{l\ddot{\theta}_2}{2} = -\frac{3g\theta_1}{2}$$

$$\frac{\partial L}{\partial \theta_2} = -\frac{mgl\theta_2}{2}, \text{ and } \frac{\partial L}{\partial \dot{\theta}_2} = \frac{ml^2\dot{\theta}_2}{3} + \frac{ml^2\dot{\theta}_1}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{\partial L}{\partial \theta_2} \Rightarrow \frac{ml^2\ddot{\theta}_2}{3} + \frac{ml^2\ddot{\theta}_1}{2} = -\frac{mgl\theta_2}{2},$$

$$\Rightarrow \frac{l\ddot{\theta}_2}{3} + \frac{l\ddot{\theta}_1}{2} = -\frac{g\theta_2}{2},$$

($\sin\theta_1 \approx \theta_1$, $\sin\theta_2 \approx \theta_2$, $\cos(\theta_1 - \theta_2) \approx 1$, $\sin(\theta_1 - \theta_2) \approx 0$)

Or,

$$8l\ddot{\theta}_1 + 3l\ddot{\theta}_2 = -9g\theta_1 \qquad (1)$$

$$2l\ddot{\theta}_2 + 3l\ddot{\theta}_1 = -3g\theta_2 \qquad (2)$$

To determine the normal modes, let us multiply (2) by α and add it onto (1),

$$(8l + 3l\alpha)\ddot{\theta}_1 + (3l + 2l\alpha)\ddot{\theta}_2 = -9g\theta_1 - 3g\theta_2\alpha$$

For properly chosen α, the coefficient ratio of $\ddot{\theta}_1$, $\ddot{\theta}_2$ on LHS would equal to the coefficient ratio of θ_1 ,θ_2 on RHS.

$$\frac{8l+3l\alpha}{3l+2l\alpha}=\frac{-9g}{-3g\alpha} \quad \Rightarrow \quad \frac{8+3\alpha}{3+2\alpha}=\frac{3}{\alpha}$$

$$\Rightarrow \quad 3\alpha^2+2\alpha-9=0$$

$$\Rightarrow \quad \alpha_{\pm}=-\frac{1}{3}\pm\frac{2\sqrt{7}}{3}$$

$\alpha_+>0$ and $\alpha_-<0$. The vibrational frequencies of the two normal modes would than be,

$$\omega_-=\sqrt{\frac{9g}{(8+3\alpha_+)l}} \text{ , and } \omega_+=\sqrt{\frac{9g}{(8+3\alpha_-)l}}$$

(f).

$$p_1 \equiv \frac{\partial L}{\partial\dot{\theta}_1}=\frac{4ml^2\dot{\theta}_1}{3}+\frac{ml^2\cos(\theta_1-\theta_2)\dot{\theta}_2}{2}$$

$$p_2 \equiv \frac{\partial L}{\partial\dot{\theta}_2}=\frac{ml^2\dot{\theta}_2}{3}+\frac{ml^2\cos(\theta_1-\theta_2)\dot{\theta}_1}{2}$$

We can solve for $\dot{\theta}_1,\dot{\theta}_2$ as

$$\dot{\theta}_1=\frac{-12p_1+18\cos(\theta_1-\theta_2)p_2}{ml^2[-16+9\cos^2(\theta_1-\theta_2)]}$$

$$\dot{\theta}_2=\frac{-48p_2+18\cos(\theta_1-\theta_2)p_1}{ml^2[-16+9\cos^2(\theta_1-\theta_2)]}$$

(g) $\mathrm{H}\equiv \dot{\theta}_1p_1+\dot{\theta}_2p_2-L$

$$=\frac{-6p_1^2-24p_2^2+18\cos(\theta_1-\theta_2)p_1p_2}{ml^2[-16+9\cos^2(\theta_1-\theta_2)]}-\frac{3mgl\cos\theta_1}{2}-\frac{mg\cos\theta_2}{2}$$