22.51 Problem Set 5 (due Fri, Oct. 26)

1 3D Green's Function

Question: Prove that the solution to,

$$\nabla^2 g(\mathbf{x}) \quad = \quad -4\pi\delta(\mathbf{x}), \qquad \mathbf{x} \in \mathbf{R}^3,$$

is

$$g(\mathbf{x}) \quad = \quad \frac{1}{|\mathbf{x}|}.$$

Answer:

For $|\mathbf{x}| > 0$, there is,

$$\nabla \frac{1}{|\mathbf{x}|} \quad = \quad -\frac{\mathbf{x}}{|\mathbf{x}|^3},$$

and so,

$$\nabla \cdot \left(-\frac{\mathbf{x}}{|\mathbf{x}|^3}\right) \quad = \quad -\frac{\nabla \cdot \mathbf{x}}{|\mathbf{x}|^3} - \mathbf{x} \cdot \left(-\frac{3\mathbf{x}}{|\mathbf{x}|^5}\right) \quad = \quad 0.$$

However, as $|\mathbf{x}| \to 0$, the terms involved get larger and larger, so $|\mathbf{x}| = 0$ becomes a singularity.

The one and only criterion that something is a δ-function is that it is 0 everywhere beside the singularity, and the singularity is integrable with result unity. Here, because,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \nabla \cdot \left(\nabla \frac{1}{|\mathbf{x}|}\right) d^3\mathbf{x} \quad = \quad \int\int\int_R \nabla \cdot \left(\nabla \frac{1}{|\mathbf{x}|}\right) d^3\mathbf{x},$$

where the integrated volume changes from whole space to a spherical region of radius R (because the integrand is zero outside of R). We then use the divergence theorem,

$$\int\int\int_R \nabla \cdot \left(\nabla \frac{1}{|\mathbf{x}|}\right) d^3\mathbf{x} \quad = \quad \int\int_S \left(\nabla \frac{1}{|\mathbf{x}|}\right) \cdot \mathbf{n} dS \quad = \quad \int\int_S -\frac{\mathbf{R}}{R^3} \cdot \mathbf{n} dS \quad = \quad -4\pi.$$

Thus, $\nabla^2(|\mathbf{x}|^{-1})$ is zero when $|\mathbf{x}| > 0$, but its volume integral gives -4π, so it can only be a δ-function.

2 Spatial-Temporal Green's Function

Question: Prove that the solution to,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{x},t) = -4\pi\rho(\mathbf{x},t), \quad \mathbf{x} \in \mathbf{R}^3,$$

is

$$\phi(\mathbf{x},t) = \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}.$$

Answer: For $|\mathbf{x}-\mathbf{x}'| > 0$, there is,

$$\nabla\left(\frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) = -\frac{\rho(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} - \frac{\rho_t(\mathbf{x}-\mathbf{x}')}{c|\mathbf{x}-\mathbf{x}'|^2}.$$

From Problem 1 we see that $\nabla \cdot (-(\mathbf{x}-\mathbf{x}')/|\mathbf{x}-\mathbf{x}'|^3) = 0$ for $|\mathbf{x}-\mathbf{x}'| > 0$, thus,

$$\begin{aligned}
&\nabla^2\left(\frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) \\
= \;& \frac{\rho_t(\mathbf{x}-\mathbf{x}')}{c|\mathbf{x}-\mathbf{x}'|} \cdot \frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} + \frac{\rho_{tt}(\mathbf{x}-\mathbf{x}')}{c|\mathbf{x}-\mathbf{x}'|} \cdot \frac{(\mathbf{x}-\mathbf{x}')}{c|\mathbf{x}-\mathbf{x}'|^2} - \\
& \frac{\rho_t}{c}\left(\frac{\nabla\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^2} - 2\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^4}\cdot(\mathbf{x}-\mathbf{x}')\right) \\
= \;& \frac{\rho_{tt}}{c^2|\mathbf{x}-\mathbf{x}'|}. \qquad (1)
\end{aligned}$$

Thus, when $|\mathbf{x}-\mathbf{x}'| > 0$,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) = 0.$$

Therefore,

$$\begin{aligned}
\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{x},t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3\mathbf{x}' \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) \\
&= \int\int\int_R d^3\mathbf{x}' \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)}{|\mathbf{x}-\mathbf{x}'|}\right) \qquad (2)
\end{aligned}$$

Since we are free to choose any R, we can choose it to be very small, so as $\mathbf{x}'$ approaches $\mathbf{x}$, we can expand $\rho(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c)$ as,

$$\rho\left(\mathbf{x}', t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right) \approx \rho(\mathbf{x}', t) - \frac{\rho_t}{c}|\mathbf{x}-\mathbf{x}'| + \mathcal{O}\left(|\mathbf{x}-\mathbf{x}'|^2\right).$$

Therefore,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{x}, t) \approx \int\int\int_R d^3\mathbf{x}' \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\rho(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|} - \frac{\rho_t}{c} + \mathcal{O}(R)\right).$$

and the integral contributions from all terms except $\nabla^2\left(\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\right)$ which is singular at $\mathbf{x} = \mathbf{x}'$ become negligible. So we have,

$$\int\int\int_R d^3\mathbf{x}'\nabla^2\left(\frac{\rho(\mathbf{x}', t)}{|\mathbf{x}-\mathbf{x}'|}\right) = \int\int\int_R d^3\mathbf{x}'\rho(\mathbf{x}', t)(-4\pi\delta(\mathbf{x}-\mathbf{x}')) = -4\pi\rho(\mathbf{x}, t),$$

QED.

3 Lorentz Transformation

Question: For observer A, any event can be labeled by (x, t). For observer B, the *same* event is labeled by (x', t'). Suppose there is a linear connection between (x, t) and (x', t'),

$$x' = \alpha x + \beta t, \quad t' = \mu x + \eta t,$$

which is based on the belief that space-time is homogeneous, where α, β, μ, η are constants, our goal is to determine α, β, μ, η.

The first condition is that B is seen by A as moving with uniform velocity v, therefore an event $(0, t')$ for B - which is how B labels himself, is labeled by A as $(x, t) = (vt, t)$. Conversely, $(0, t)$ for A should be considered $(-vt', t')$ for B.

The second condition is that the speed of light is c for both A and B, therefore $(x, t) = (ct, t)$ corresponds to $(x', t') = (ct', t')$.

Lastly, the space should be isotropic, so using $-x$ as labeling variable should be no different from using x as labeling variable. This suggests that if α is a function of v, then $\alpha(v) = \alpha(-v)$.

Please solve for $\alpha(v), \beta(v), \mu(v), \eta(v)$.

Answer: The first condition gives,

$$\alpha(vt) + \beta t \quad = \quad 0,$$

so $\beta = -\alpha v$. For the reciprocal case, the following is a useful identity to remember,

$$\begin{pmatrix} \alpha & \beta \\ \mu & \eta \end{pmatrix}^{-1} \quad = \quad \frac{1}{\alpha\eta - \mu\beta} \begin{pmatrix} \eta & -\beta \\ -\mu & \alpha \end{pmatrix},$$

and so,

$$x = \frac{\eta x' - \beta t'}{\alpha\eta - \mu\beta}, \qquad t = \frac{-\mu x' + \alpha t'}{\alpha\eta - \mu\beta}. \tag{3}$$

Thus,

$$\eta(-vt') - \beta t' \quad = \quad 0,$$

so $\beta = -\eta v$, therefore $\beta = -\alpha v = -\eta v$ and $\alpha = \eta$.

From the second condition, we have,

$$ct' \quad = \quad \alpha ct - \alpha vt, \qquad t' \quad = \quad \mu ct + \alpha t,$$

so,

$$c \quad = \quad \frac{c - v}{c\mu/\alpha + 1} \qquad \longrightarrow \qquad \mu/\alpha = -v/c^2.$$

For the third requirement, from (3) we see that,

$$\alpha(-v) \quad = \quad \frac{\eta(v)}{\alpha(v)\eta(v) - \mu(v)\beta(v)} \quad = \quad \frac{\alpha(v)}{\alpha(v)\eta(v) - \mu(v)\beta(v)},$$

if $\alpha(v) = \alpha(-v)$, we must have,

$$1 \quad = \quad \alpha(v)\eta(v) - \mu(v)\beta(v) \quad = \quad \alpha^2 - \alpha(-v/c^2)(-\alpha v),$$

which means that,

$$\alpha(v) \quad = \quad \frac{1}{\sqrt{1 - v^2/c^2}},$$

and so,

$$x' \quad = \quad \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' \quad = \quad \frac{-vx/c^2 + t}{\sqrt{1 - v^2/c^2}}, \qquad x \quad = \quad \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}, \quad t \quad = \quad \frac{vx'/c^2 + t'}{\sqrt{1 - v^2/c^2}}.$$

4 Doppler Shift

Question: A wave is characterized by $A\exp(ikx - i\omega t)$. In a different frame, it must also be characterized by $A'\exp(ik'x' - i\omega' t')$. A and A' can be very different for various reasons, but it is unlikely that the phase, $\theta \equiv kx - \omega t$, differs from the phase, $\theta' \equiv k'x' - \omega' t'$, because it would be a very strange world if a wave-crest event in one frame is not a wave-crest event in the other frame.

Assuming $\theta = \theta'$, so θ is a *frame invariant*, derive the transformation law from (k, ω) to (k', ω') between the two inertial frames described in Problem 3. Specialize the result to when $\omega/k = c$, and show that ω'/k' is still c, even though ω' differs from ω.

Doppler shift of spectral lines is the main method to measure the relative speed between here and distant stars.

Answer: If, as Problem 3 shows,

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{-vx/c^2 + t}{\sqrt{1 - v^2/c^2}},$$

then

$$k'x' - \omega' t' = \frac{k'x - k'vt}{\sqrt{1 - v^2/c^2}} - \frac{-\omega' vx/c^2 + \omega' t}{\sqrt{1 - v^2/c^2}},$$

which if it must agree with $kx - \omega t$ for any (x, t), can only happen if,

$$k = \frac{k' + \omega' v/c^2}{\sqrt{1 - v^2/c^2}}, \quad \omega = \frac{\omega' + k'v}{\sqrt{1 - v^2/c^2}},$$

or conversely,

$$k' = \frac{k - \omega v/c^2}{\sqrt{1 - v^2/c^2}}, \quad \omega' = \frac{\omega - kv}{\sqrt{1 - v^2/c^2}}.$$

$\omega' = \omega - kv$ is the classical Doppler shift formula, applicable to small v. The $(1 - v^2/c^2)^{-1/2}$ factor is the relativistic correction. Since,

$$\frac{\omega}{k} = \frac{\omega' + k'v}{k' + \omega' v/c^2} = \frac{\omega'/k' + v}{1 + (\omega'/k')(v/c^2)},$$

it is easy to see that if $\omega'/k' = c$, then $\omega/k = c$.