# 22.51 Problem Set 7 (due Fri, Nov. 9)

## 1 Form Factor of Ellipsoid

**Question**: We showed that the form factor of a dielectric sphere is,

$$F(\mathbf{Q}) \quad = \quad \frac{k_0^2(m^2-1)}{4\pi}\cdot\frac{4\pi a^3}{3}\cdot\frac{3j_1(|\mathbf{Q}|a)}{|\mathbf{Q}|a}.$$

where $\mathbf{Q}\equiv\mathbf{k}_0-\mathbf{k}_0'$ is the scattering vector. Based on this, is it straightforward to get the form factor of an ellipsoid with principal axis lengths $a$,$b$,$c$?

**Answer**: Let us first rotate to a frame where $x$ is along the $a$-axis of the ellipsoid, $y$ is along the $b$-axis, and $z$ is along the $c$-axis, e.g., define,

$$\mathbf{Q}'\equiv\left(Q_x',Q_y',Q_z'\right),\quad Q_x'\equiv\mathbf{Q}\cdot\frac{\mathbf{a}}{|\mathbf{a}|},\ \ Q_y'\equiv\mathbf{Q}\cdot\frac{\mathbf{b}}{|\mathbf{b}|},\ \ Q_z'\equiv\mathbf{Q}\cdot\frac{\mathbf{c}}{|\mathbf{c}|},\tag{1}$$

from any initial frame. Then, we make the observation that,

$$\begin{aligned}
& \int_{\text{ellipsoid}}\exp\left(i\mathbf{Q}'\cdot\mathbf{x}\right)d^3\mathbf{x} \\
= & \int_{\text{ellipsoid}}\exp\left(i(Q_x'x+Q_y'y+Q_z'z)\right)dxdydz \\
= & \frac{bc}{a^2}\int_{\text{ellipsoid}}\exp\left(i(Q_x'x+Q_y'\frac{b}{a}\cdot\frac{a}{b}y+Q_z'\frac{c}{a}\cdot\frac{a}{c}z)\right)dx\left(\frac{a}{b}dy\right)\left(\frac{a}{c}dz\right) \\
= & \frac{bc}{a^2}\int_{\text{sphere}}\exp\left(i(Q_x''x+Q_y''y+Q_z''z)\right)dxdydz,
\end{aligned}\tag{2}$$

where the sphere has radius $a$, and,

$$\mathbf{Q}'' \quad \equiv \quad \left(Q_x',\ \ Q_y'\frac{b}{a},\ \ Q_z'\frac{c}{a}\right).\tag{3}$$

However, the last expression of (2) is identical to the form factor of a sphere, except by a constant Jacobian factor $bc/a^2$. Therefore, the form factor for an ellipsoid is,

$$F(\mathbf{Q}) \quad = \quad \frac{k_0^2(m^2-1)}{4\pi}\cdot\frac{4\pi abc}{3}\cdot\frac{3j_1(|\mathbf{Q}''|a)}{|\mathbf{Q}''|a},$$

where $\mathbf{Q}''$ is a simple function of $\mathbf{Q}$ via (1) and (3).

The point of this exercise is not to re-work the math but to illustrate the power of an invariant transformation on $\mathbf{Q} \cdot \mathbf{x}$. Suppose someone has worked out the integral of,

$$\int_{\text{sphere}} \tanh^3(\mathbf{Q} \cdot \mathbf{x}) \sin(\mathbf{Q} \cdot \mathbf{x}) (\mathbf{Q} \cdot \mathbf{x})^5 d^3\mathbf{x},$$

it is *as easy* to get the result for an ellipsoid as above. All depends on the *symmetry* properties of $\mathbf{Q} \cdot \mathbf{x}$.

## 2 Static Polarizability of Dielectric Sphere

**Question**: Prove Equation (4.180) and explain its significance in connecting radiation with scattering phenomena.

**Answer**: The governing equations for $\mathbf{E}(\mathbf{x})$ in a linear-response dielectric medium, when there is no $\mathbf{B}$ field involved, are shown in class to be,

$$\nabla \cdot (\epsilon \mathbf{E}) = 0, \quad \nabla \times \mathbf{E} = 0,$$

where the dielectric constant $\epsilon$ may depend on $\mathbf{x}$. In our problem,

$$\epsilon(\mathbf{x}) = \begin{cases} \epsilon_1, & |\mathbf{x}| < a, \\ \epsilon_0, & |\mathbf{x}| \geq a, \end{cases} \tag{4}$$

Since $\nabla \times \mathbf{E} = 0$, using existence theorem we know there exists scalar field $\phi(\mathbf{x})$ such that,

$$\mathbf{E}(\mathbf{x}) = -\nabla \phi(\mathbf{x}), \tag{5}$$

and $\phi(\mathbf{x})$ should be continuous across the interface (even though its first-order derivative may not). When we plug (4) and (5) into $\nabla \cdot (\epsilon \mathbf{E}) = 0$, the condition turns out to be $\nabla^2 \phi = 0$ both inside and outside of the sphere, and $\epsilon_1 \partial_n \phi$ inside should match $\epsilon_0 \partial_n \phi$ outside across the interface. Furthermore, if the external field is $E_0 \mathbf{e}_z$, then the *leading* $x \to \infty$ *asymptote* of $\phi(\mathbf{x})$ should be $-E_0 r \cos\theta$.

Because there is no azimuthal variation in the setup, $\phi(\mathbf{x})$ should take the form,

$$\phi(\mathbf{x}) = g(r) f(\cos\theta),$$

both inside and outside of the sphere. Since,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\cos\theta}\left((1-\cos^2\theta)\frac{\partial}{\partial\cos\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

and let us furthermore assume the angular dependence is simply,

$$f(\cos\theta) = \cos\theta,$$

then the equation for $g(r)$ would be,

$$\frac{d}{dr}\left(r^2 g'(r)\right) - 2g(r) = 0,$$

whose possible solutions are,

$$g(r) = r, \ \ r^{-2}.$$

For $\phi(\mathbf{x})$ inside the sphere, the $r^{-2}$ solution is impossible, therefore,

$$\phi(\mathbf{x}) = Ar\cos\theta, \quad |\mathbf{x}| < a.$$

For $\phi(\mathbf{x})$ outside of the sphere, the solution may look like,

$$\phi(\mathbf{x}) = -E_0 r\cos\theta + Br^{-2}\cos\theta, \quad |\mathbf{x}| \geq a.$$

To satisfy the continuity condition, we have,

$$A = -E_0 + Ba^{-3}.$$

To satisfy the $\mathbf{D}(\mathbf{x})$ divergence-free condition, we must have,

$$\epsilon_1 A = \epsilon_0(-E_0 - 2Ba^{-3}).$$

So there must be,

$$m^2 A = -E_0 - 2(A + E_0),$$

therefore,

$$A = -\frac{3E_0}{2+m^2}, \quad \phi(\mathbf{x}) = -\frac{3E_0}{2+m^2} r\cos\theta, \quad |\mathbf{x}| < a.$$

therefore the electric field inside the sphere is,

$$\mathbf{E} \;=\; \frac{3\mathbf{E}_0}{2+m^2},$$

and the polarization density is,

$$\mathbf{P} \;=\; \frac{\mathbf{D}-\mathbf{E}}{4\pi} \;=\; \frac{\epsilon_1-1}{4\pi}\mathbf{E} \;=\; \frac{\epsilon_1-1}{4\pi}\frac{3\mathbf{E}_0}{2+m^2},$$

so the total induced dipole of the sphere is,

$$\mathbf{p} \;=\; \frac{4\pi a^3}{3}\mathbf{P} \;=\; a^3\frac{\epsilon_1-1}{2+m^2}\mathbf{E}_0.$$

When the background is vacuum, $\epsilon_0 = 1$, $\epsilon_1 = m^2$, equation (4.180) is correct. When the background is not vacuum, one must subtract off the background polarization density from $\mathbf{P}$, because there still would be no scattering in a *homogeneous* dielectric medium $\epsilon_1 = \epsilon_0 > 1$, even when there *is* finite and time-varying polarization. In other words, the scattering is really caused by the *extra* polarization density inside the sphere. When one works out the details, equation (4.180) would still be correct. But instead, we again have a simpler approach using a transformation, where we map the background to vacuum!

The idea is the following. Suppose we want to solve,

$$\nabla\times\mathbf{E} \;=\; -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t},\quad \nabla\times\frac{\mathbf{B}}{\mu(\mathbf{x})} \;=\; \frac{1}{c}\frac{\partial\epsilon(\mathbf{x})\mathbf{E}}{\partial t},\quad \nabla\cdot\mathbf{B} \;=\; 0,\quad \nabla\cdot\epsilon(\mathbf{x})\mathbf{E} \;=\; 0, \qquad (6)$$

in the case of $\mu(\mathbf{x}) \equiv 1$, and $\epsilon(\mathbf{x})$ may vary with space but with baseline value $\epsilon_0$, then by *redefining space and magnetic field*, we can map to a world spatially larger than ours, with background dielectric constant 1! Specifically, Let us define $\mathbf{x}'$ such that,

$$\nabla \;\equiv\; \sqrt{\epsilon_0}\nabla',\quad \mathbf{B} \;\equiv\; \sqrt{\epsilon_0}\mathbf{B}',\quad \mathbf{E}' \;\equiv\; \mathbf{E},$$

then we can convince ourselves that,

$$\nabla'\times\mathbf{E}' \;=\; -\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t},\quad \nabla'\times\frac{\mathbf{B}'}{\mu(\mathbf{x}')} \;=\; \frac{1}{c}\frac{\partial\epsilon(\mathbf{x}')/\epsilon_0\mathbf{E}'}{\partial t},\quad \nabla'\cdot\mathbf{B}' \;=\; 0,\quad \nabla'\cdot\epsilon(\mathbf{x})/\epsilon_0\mathbf{E}' \;=\; 0.$$

In other words, $\{\mathbf{E}'(\mathbf{x}'),\mathbf{B}'(\mathbf{x}')\}$ satisfy Maxwell's equations as if the background *is* vacuum. Therefore, in this parallel world, the solution is equation (4.177) but with $\mathbf{E}_s \to \mathbf{E}'_s$, $k \to k'/\sqrt{\epsilon_0}$, $r \to r' = \sqrt{\epsilon_0}r$, $a \to a' = \sqrt{\epsilon_0}a$. We find that because the magnitude of $\mathbf{E}$ is not influenced before and after the transformation, and because $k$, $r$, $a$ are influenced but their combined effect cancel, equation (4.177) is still correct.

## 3 Differential and Total Scattering Cross-Sections

**Question**:

**a**. Fill in the missing details about the far-field magnetic field $\mathbf{B}(\mathbf{x}, t)$ and the Poynting vector $\mathbf{S}(\mathbf{x}, t)$, that enables one to derive Equation (4.184).

**Answer**: At far-field, the scattered wave locally is like a plane-wave with wave-vector $k_0\mathbf{e}_r$, therefore from equation (4.150),

$$\mathbf{B}_s(\mathbf{x}, t) = \frac{c}{i\omega}(ik_0\mathbf{e}_r) \times \mathbf{E}_s(\mathbf{x}, t) = \frac{c}{v_0}\mathbf{e}_r \times \mathbf{E}_s(\mathbf{x}, t) = n_0\mathbf{e}_r \times \mathbf{E}_s(\mathbf{x}, t),$$

and so,

$$\mathbf{S}_s(\mathbf{x}, t) = \frac{c}{4\pi}(\mathbf{E}_s \times \mathbf{B}_s) = \frac{cn_0}{4\pi}|\mathbf{E}_s(\mathbf{x}, t)|^2\mathbf{e}_r.$$

The incoming energy flux is,

$$\mathbf{S}(\mathbf{x}, t) = \frac{cn_0}{4\pi}|\mathbf{E}_0(\mathbf{x}, t)|^2\mathbf{e}_{k_0}.$$

Therefore the differential scattering cross-section is indeed,

$$\frac{d\sigma}{d\Omega} = \lim_{r\to\infty}\frac{|\mathbf{S}_s(\mathbf{x}, t)|r^2}{|\mathbf{S}(\mathbf{x}, t)|} = \langle|\mathbf{f}(\theta)|^2\rangle,$$

as in equation (4.183).

A paradox can be posed as the following. Why should not

$$\mathbf{B}_s(\mathbf{x}, t) = \mathbf{g}(\theta)\frac{e^{i(kr-\omega t)}}{r}B_0, \quad \mathbf{g}(\theta) \equiv k_0^2\alpha(\mathbf{I} - \mathbf{e}_r\mathbf{e}_r) \cdot \frac{\mathbf{B}_0}{B_0},$$

where $\mathbf{g}(\theta)$ is clearly different from $\mathbf{f}(\theta)$, since $\mathbf{B}$ appears to be symmetric with $\mathbf{E}$ in that they both satisfy,

$$-\nabla^2\mathbf{W} - n^2k_v^2\mathbf{W} = 0,$$

in a homogeneous medium, so we can use the same perturbation theory?

The answer is: it is true that $\mathbf{B}$ and $\mathbf{E}$ have symmetric "positions" in a *homogeneous medium.*

But when $\mu = \mu(\mathbf{x})$, $\epsilon = \epsilon(\mathbf{x})$, their positions are not necessarily symmetric. A careful examination of equation (4.153) derived from (6) shows that it is only true for $\mathbf{E}$ when $\mu(\mathbf{x}) \equiv 1$, and not true for $\mathbf{B}$. Therefore, $\mathbf{B}$ is puppet to $\mathbf{E}$ when $\mu(\mathbf{x}) \equiv 1$ everywhere, and vice versa.

**b**. Equation (4.185) is clearly wrong. Explain why and derive the correct result. Does it change the final conclusion, though?

**Answer**: Let,

$$\mathbf{E}_0 \;=\; E_0\mathbf{e}_z \;=\; E_0(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta),$$

therefore,

$$(\mathbf{I} - \mathbf{e}_r\mathbf{e}_r)\cdot\frac{\mathbf{E}_0}{E_0} \;=\; -\mathbf{e}_\theta\sin\theta,$$

and $\sin\theta = 1$ only if $\theta = \pi/2$, in the VV plane. Therefore, the (4.185) estimate should be scaled by a factor,

$$\frac{1}{4\pi}\int d\Omega\sin^2\theta \;=\; \frac{1}{4\pi}\int 2\pi d\cos\theta(1-\cos^2\theta) \;=\; \frac{2}{3}.$$

But this would not change the conclusion that a small dielectric particle is much less reflective than its size suggests when $kd \ll 1$.

# 4 Why is the Sky Blue?

**Question**: Let $f(\omega)d\omega$ be the probability that a measurement of sun's radiation on moon yields a photon sample within frequency range $(\omega, \omega + d\omega)$. Show why our sky is blue - or, to be more precise, "bluer" than the sun.

**Answer**: On moon, when one looks directly at the sun, the probability distribution of photon frequency is,

$$P_1(\omega) \;=\; f(\omega).$$

On earth, when we look at the sky (away from the sun's direction), the photons that come into our eyes must have undergone scattering by the air molecules in the atmosphere. Assuming just single scattering events occur, the probability distribution is,

$$P_2(\omega) \;=\; \frac{f(\omega)\omega^4}{\int_0^\infty f(\omega)\omega^4 d\omega},$$

since the scattering cross-section is $\propto \omega^4$ in any direction. $P_2(\omega)$ distribution is clearly going to have a larger mean than the $P_1(\omega)$ distribution. Therefore the sky looks "blue".

When we do look directly at the sun on earth, the probability distribution is,

$$P_3(\omega) \quad = \quad \frac{P_1(\omega) - rP_2(\omega)}{1-r},$$

where $r$ is the number ratio of scattered photons when the sunlight pierce the atmosphere. $P_3(\omega)$ is going to look even "redder" than $P_1(\omega)$. The larger $r$ is, the more it is so. Therefore the sun looks especially red at sunrise and sunset, since when the sun is near the horizon it is separated from our eyes by the thickest layer of air!

Could this redshift spell trouble for astronomers when they, say, measure the *real* redshift of stars, since anything that depends on air thickness is going to terribly tricky?

When we think about it, the answer is no. The spectral lines themselves do not shift due to the scattering loss, but only their relative weights change, causing the overall redshift. Therefore this scattering 'redshift' is of a nature entirely different from the Doppler redshift.