22.51 Quiz I (90 minutes, Chen&Kotlarchyk book only)

Question 1 (7 pt)

A heavy rod is rotating in a fixed plane, say xy plane, with constant angular frequency ω. A ball of mass m is attached to the rod and is only able to slide along the rod, so the ball's only degree of freedom is its distance to the origin, r. Ignoring friction, and assuming the ball is under the influence of a central potential $V(r)$, derive the equation of motion for $r(t)$ using Lagrangian mechanics.

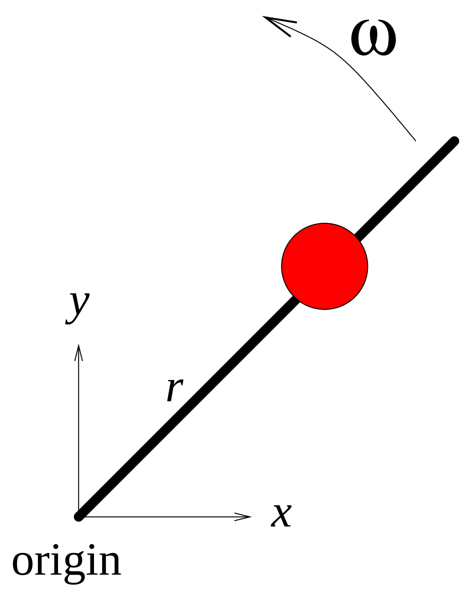

Figure 1: Ball sliding along a rotating rod of constant angular frequency ω.

Answer: Given $r(t)$, the velocity of the ball is,

$$\mathbf{v}(t) = \dot{r}\mathbf{e}_r + \omega r \mathbf{e}_\theta,$$

where $\mathbf{e}_r$,$\mathbf{e}_\theta$ are unit vectors in the polar frame, so the kinetic energy is,

$$K = \frac{m}{2}\mathbf{v}\cdot\mathbf{v} = \frac{m}{2}\left(\dot{r}^2 + \omega^2 r^2\right),$$

thus the Lagrangian is,

$$\mathcal{L}(r,\dot{r}) = \frac{m}{2}\left(\dot{r}^2 + \omega^2 r^2\right) - V(r).$$

Therefore,

$$\frac{\partial \mathcal{L}}{\partial r} = m\omega^2 r - V'(r),$$

and

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r},$$

so,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r},$$

would give us,

$$m\ddot{r} = m\omega^2 r - V'(r).$$

where $-V'(r)$ is the ordinary force if the rod is not rotating and $m\omega^2 r$ is the so-called centrifugal force.

Question 2-4 deal with quantum mechanics, in which $\hbar$ is taken to be 1.

Question 2 (7 pt)

Two particles of mass m_1 and m_2 interact with each other in 1D, and suppose their interaction is a function of their separation $x_2 - x_1$ only, then (in the Schrodinger's picture),

$$\hat{H} = -\frac{1}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{1}{2m_2}\frac{\partial^2}{\partial x_2^2} + V(x_2 - x_1), \quad |\psi\rangle = \psi(x_1, x_2, t).$$

Find a symmetry operator for the system, and show that the total momentum,

$$\hat{p}_1 + \hat{p}_2 \equiv -i\partial/\partial x_1 - i\partial/\partial x_2$$

is a conserved quantity, i.e., $\langle \hat{p}_1 + \hat{p}_2 \rangle$ is independent of time.

Answer: On class, when we study the eigenfunctions of a simple harmonic oscillator, I have introduced the inversion operator $\hat{P}$,

$$\hat{P}\psi(x) \equiv \psi(-x),$$

as an example of symmetry operators, where because the potential energy $m\omega^2 x^2/2$ and the kinetic energy operators both commute with $\hat{P}$, there is,

$$[\hat{P}, \hat{\mathcal{H}}] = 0,$$

making $\hat{P}$ a proper symmetry operator for that system. One consequence of this is that the eigenfunctions $\psi_n(x)$ of $\hat{\mathcal{H}}$ must have definite *parity*, either $+1$ or -1. Another consequence is that the measurement average of any symmetry operator that does not explicitly depend on time is time-independent, since there is,

$$\frac{d\langle \hat{P}\rangle}{dt} = \frac{1}{i\hbar}\left\langle [\hat{P},\hat{\mathcal{H}}]\right\rangle + \left\langle \frac{\partial \hat{P}}{\partial t}\right\rangle.$$

In this problem, what one needs to show is that,

$$[\hat{p}_1 + \hat{p}_2, \hat{\mathcal{H}}] = 0, \tag{1}$$

so its measurement average would not depend on time. $\hat{p}_1 + \hat{p}_2$ is then called a symmetry operator for the system, or more precisely a *symmetry operation generator* for the the system.

To prove (1), let us observe that $\partial/\partial x_1$ commutes with both $\partial^2/\partial x_1^2$ and $\partial^2/\partial x_2^2$, and the same is true for $\partial/\partial x_2$, therefore all we need to show is that,

$$\left[\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\ V\right] = 0.$$

It is easy to show that,

$$\left[\frac{\partial}{\partial x_1},\ V(x_1,x_2)\right] = \frac{\partial V}{\partial x_1},$$

and

$$\left[\frac{\partial}{\partial x_2},\ V(x_1,x_2)\right] = \frac{\partial V}{\partial x_2}.$$

If $V(x_1,x_2)$ takes the form $V(x_2 - x_1) = V(\mu)$, then

$$\frac{\partial V}{\partial x_1} = \frac{dV}{d\mu}\cdot\frac{\partial \mu}{\partial x_1} = -\frac{dV}{d\mu},$$

$$\frac{\partial V}{\partial x_2} = \frac{dV}{d\mu}\cdot\frac{\partial \mu}{\partial x_2} = \frac{dV}{d\mu},$$

and they cancel if summed.

Question 3 (6 pt)

Find 2×2 matrix representations for $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ under the basis set $\{|\psi_1\rangle, |\psi_2\rangle\}$,

$$|\psi_1\rangle \;=\; \left|\frac{1}{2},\frac{1}{2}\right\rangle, \quad |\psi_2\rangle \;=\; \left|\frac{1}{2},-\frac{1}{2}\right\rangle,$$

where,

$$\hat{J}^2\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle = \frac{1}{2}\frac{3}{2}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle, \quad \hat{J}_z\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle = \pm\frac{1}{2}\left|\frac{1}{2},\pm\frac{1}{2}\right\rangle.$$

As a check of your result, you may verify that these 2×2 *matrices*, $\mathbf{J}_x$, $\mathbf{J}_y$, $\mathbf{J}_z$, satisfy the fundamental relations,

$$[\mathbf{J}_i,\ \mathbf{J}_j] \;=\; i\epsilon_{ijk}\mathbf{J}_k.$$

Answer: There is,

$$\hat{J}_+\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)-\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)}\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle, \quad \hat{J}_+\left|\frac{1}{2},\frac{1}{2}\right\rangle = 0,$$

and,

$$\hat{J}_-\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle, \quad \hat{J}_-\left|\frac{1}{2},-\frac{1}{2}\right\rangle = 0.$$

therefore $\hat{J}_+$ and $\hat{J}_-$ operators are closed within $\left\{\left|\frac{1}{2},\frac{1}{2}\right\rangle,\left|\frac{1}{2},-\frac{1}{2}\right\rangle\right\}$ basis set, and their matrix representations are,

$$\mathbf{J}_+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{J}_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since,

$$\hat{J}_+ \;\equiv\; \hat{J}_x + i\hat{J}_y, \quad \hat{J}_- \;\equiv\; \hat{J}_x - i\hat{J}_y,$$

we have,

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right), \quad \hat{J}_y = \frac{1}{2i}\left(\hat{J}_+ - \hat{J}_-\right),$$

therefore,

$$\mathbf{J}_x \;=\; \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad \mathbf{J}_y \;=\; \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, \quad \mathbf{J}_z \;=\; \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Bonus Question 4 (7 pt)

A 1D free-particle of mass 1 can be described by $\psi(x,t)$,

$$-\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi(x,t) \quad = \quad i\frac{\partial}{\partial t}\psi(x,t),$$

which is related to its momentum representation $\phi(k,t)$ as,

$$\psi(x,t) \quad = \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k,t)\exp(ikx)dk.$$

Suppose,

$$\phi(k,0) \quad = \quad \left(\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(k-k_0)^2}{2\sigma^2}\right)\right)^{1/2}, \quad k_0 \in \mathbf{R}.$$

Solve for $\psi(x,t)$, and explain under what conditions of k_0, σ and t can we consider $\psi(x,t)$ as representing a classical particle moving with speed k_0.

Answer: $\phi(k,t)$ satisfies,

$$-i\frac{k^2}{2}\phi(k,t) \quad = \quad \frac{\partial}{\partial t}\phi(k,t),$$

whose solution is,

$$\phi(k,t) \quad = \quad \exp\left(-\frac{ik^2t}{2}\right)\phi(k,0),$$

therefore,

$$\phi(k,t) \quad = \quad \frac{1}{(2\pi\sigma^2)^{1/4}}\exp\left(-\frac{(k-k_0)^2}{4\sigma^2}-\frac{ik^2t}{2}\right),$$

and so,

$$\psi(x,t) \quad = \quad \frac{1}{\sqrt{2\pi}}\frac{1}{(2\pi\sigma^2)^{1/4}}\int_{-\infty}^{\infty}\exp\left(-\frac{(k-k_0)^2}{4\sigma^2}-\frac{ik^2t}{2}+ikx\right)dk.$$

Because there is,

$$\begin{aligned}
& -(k-k_0)^2 - 2i\sigma^2k^2t + i4\sigma^2kx \\
= \quad & -k^2 + 2k_0k - k_0^2 - 2it\sigma^2k^2 + 4i\sigma^2xk \\
= \quad & -(1+2it\sigma^2)k^2 + (2k_0+4i\sigma^2x)k - k_0^2 \\
= \quad & -(1+2it\sigma^2)\left(k^2 - \frac{2k_0+4i\sigma^2x}{1+2it\sigma^2}k\right) - k_0^2 \\
= \quad & -(1+2it\sigma^2)\left(k^2 - \frac{2k_0+4i\sigma^2x}{1+2it\sigma^2}k + \left(\frac{k_0+2i\sigma^2x}{1+2it\sigma^2}\right)^2\right) \\
& +(1+2it\sigma^2)\left(\frac{k_0+2i\sigma^2x}{1+2it\sigma^2}\right)^2 - k_0^2
\end{aligned}$$

$$
\begin{aligned}
= \quad & -(1+2it\sigma^2)\left(k-\frac{k_0+2i\sigma^2 x}{1+2it\sigma^2}\right)^2 + \\
& (1+2it\sigma^2)\left(\frac{k_0+2i\sigma^2 x}{1+2it\sigma^2}\right)^2 - k_0^2.
\end{aligned}
$$

Thus, by completing the Gaussian integral, we have,

$$
\psi(x,t) \;=\; \frac{1}{(2\pi\sigma^2)^{1/4}}\sqrt{\frac{2\sigma^2}{1+2it\sigma^2}}\exp\left(\frac{(k_0+2i\sigma^2 x)^2}{4\sigma^2(1+2it\sigma^2)}-\frac{k_0^2}{4\sigma^2}\right).
$$

Thus,

$$
\begin{aligned}
|\psi(x,t)|^2 = \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{(k_0+2i\sigma^2 x)^2}{4\sigma^2(1+2it\sigma^2)}+\frac{(k_0-2i\sigma^2 x)^2}{4\sigma^2(1-2it\sigma^2)}-\frac{k_0^2}{2\sigma^2}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{2\mathrm{Re}\left[(k_0+2i\sigma^2 x)^2(1-2it\sigma^2)\right]}{4\sigma^2(1+4t^2\sigma^4)}-\frac{k_0^2}{2\sigma^2}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{k_0^2-4\sigma^4x^2+8k_0\sigma^4xt\sigma^2}{2\sigma^2(1+4t^2\sigma^4)}-\frac{k_0^2}{2\sigma^2}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{k_0^2-4\sigma^4x^2+8k_0\sigma^4xt-k_0^2-4t^2\sigma^2k_0^2}{2\sigma^2(1+4t^2\sigma^4)}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{-4\sigma^4(x^2-2k_0xt+k_0^2t^2)}{2\sigma^2(1+4t^2\sigma^4)}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{-2\sigma^2(x^2-2k_0xt+k_0^2t^2)}{1+4t^2\sigma^4}\right)\\
= \quad & \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\exp\left(\frac{-2\sigma^2(x-k_0t)^2}{1+4t^2\sigma^4}\right), \qquad (2)
\end{aligned}
$$

and one can verify that,

$$
\int_{-\infty}^{\infty}|\psi(x,t)|^2\,dx \;=\; \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{2\sigma^2}{\sqrt{1+4t^2\sigma^4}}\sqrt{2\pi\frac{1+4t^2\sigma^4}{4\sigma^2}} \;=\; 1.
$$

Because the maximal probability occurs at $x = k_0 t$ always, one can consider $\psi(x,t)$ as representing a classical particle of speed k_0 (the group speed) as long as the wave-pack width,

$$
\Delta x \;\equiv\; \frac{\sqrt{1+4t^2\sigma^4}}{2\sigma},
$$

is still microscopic.