22.51 Quiz II (90 min, Chen&Kotlarchyk book only)

Question 1 (7 pt)

Solve for the <u>far-field</u> $\mathbf{E}(\mathbf{x},t)$, $\mathbf{B}(\mathbf{x},t)$, $\mathbf{S}(\mathbf{x},t)$ radiation of an oscillating magnetic dipole,

$$\mathbf{M}(\mathbf{x},t) \;=\; m_0\cos(\omega t)\mathbf{e}_z\delta(\mathbf{x}).$$

and calculate the time-averaged radiation power.

Answer: Define $m(t) \equiv m_0\cos(\omega t)$, and retarded field

$$[m] \;\equiv\; m\left(t-\frac{r}{c}\right),$$

which satisfies,

$$\partial_r[m] \;=\; -\frac{[\dot{m}]}{c}, \quad \partial_\theta[m] = \partial_\phi[m] = 0.$$

The magnetic Hertz vector is,

$$\Pi_m(\mathbf{x},t) \;=\; \frac{[m]}{r}\mathbf{e}_z \;=\; \frac{[m]}{r}(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta).$$

We have,

$$\nabla \;=\; \mathbf{e}_r\partial_r + \frac{\mathbf{e}_\theta}{r}\partial_\theta + \frac{\mathbf{e}_\phi}{r\sin\theta}\partial_\phi,$$

therefore,

$$\mathbf{A} \;=\; \nabla\times\Pi_m \;=\; \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\phi \\ \partial_r & r^{-1}\partial_\theta & (r\sin\theta)^{-1}\partial_\phi \\ [m]\cos\theta/r & -[m]\sin\theta/r & 0 \end{vmatrix} \;=\; \mathbf{e}_\phi\left[\frac{[\dot{m}]}{cr}\sin\theta + \mathcal{O}\left(\frac{1}{r^2}\right)\right],$$

and so,

$$\mathbf{E}(\mathbf{x},t) \;=\; -\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x},t)}{\partial t} \;=\; -\mathbf{e}_\phi\frac{[\ddot{m}]}{c^2 r}\sin\theta,$$

and,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\phi \\ \partial_r & r^{-1}\partial_\theta & (r\sin\theta)^{-1}\partial_\phi \\ 0 & 0 & [\dot{m}]\sin\theta/cr \end{vmatrix} \;=\; \mathbf{e}_\theta\frac{[\ddot{m}]}{c^2 r}\sin\theta + \mathcal{O}\left(\frac{1}{r^2}\right),$$

therefore the Poynting vector is,

$$\mathbf{S}(\mathbf{x},t) \;=\; \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t) \;\approx\; -\frac{c}{4\pi}\left(\frac{[\ddot{m}]}{c^2 r}\sin\theta\right)^2 \mathbf{e}_\phi\times\mathbf{e}_\theta \;=\; \frac{c}{4\pi}\left(\frac{[\ddot{m}]}{c^2 r}\sin\theta\right)^2 \mathbf{e}_r,$$

and the total time-averaged radiation power is,

$$P \;=\; \frac{1}{2}\cdot\int r^2 d\Omega \frac{m_0^2\omega^4}{4\pi c^3 r^2}\sin^2\theta \;=\; \frac{m_0^2\omega^4}{3c^3}.$$

Question 2 (6 pt)

Demonstrate in what sense the Poynting vectors from *two* harmonically oscillating radiation sources (such as that of Problem 1) of *different* frequencies $\omega_1 \neq \omega_2$, can be added directly.

Answer: In the time-averaged sense. Let the radiation fields from source 1 be $\mathbf{E}_1(\mathbf{x},t)$, $\mathbf{B}_1(\mathbf{x},t)$, and the radiation fields from source 2 be $\mathbf{E}_2(\mathbf{x},t)$, $\mathbf{B}_2(\mathbf{x},t)$. Since the Maxwell's equations are *linear*, the correct solution when both sources are present is,

$$\mathbf{E}(\mathbf{x},t) \;=\; \mathbf{E}_1(\mathbf{x},t)+\mathbf{E}_2(\mathbf{x},t), \quad \mathbf{B}(\mathbf{x},t) \;=\; \mathbf{B}_1(\mathbf{x},t)+\mathbf{B}_2(\mathbf{x},t),$$

therefore the Poynting vector is,

$$\mathbf{S}(\mathbf{x},t) \;=\; \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \;=\; \mathbf{S}_1(\mathbf{x},t)+\mathbf{S}_2(\mathbf{x},t)+\frac{c}{4\pi}\mathbf{E}_1\times\mathbf{B}_2+\frac{c}{4\pi}\mathbf{E}_2\times\mathbf{B}_1,$$

where,

$$\mathbf{S}_1(\mathbf{x},t) \;\equiv\; \frac{c}{4\pi}\mathbf{E}_1(\mathbf{x},t)\times\mathbf{B}_1(\mathbf{x},t), \quad \mathbf{S}_2(\mathbf{x},t) \;\equiv\; \frac{c}{4\pi}\mathbf{E}_2(\mathbf{x},t)\times\mathbf{B}_2(\mathbf{x},t),$$

are the Poynting vectors when the other source is not present. Therefore we see that the Poynting vectors are not directly additive, at least not in the instantaneous sense. However, we note that $\mathbf{E}_1(\mathbf{x},t)$ and $\mathbf{E}_2(\mathbf{x},t)$ oscillate temporally in frequencies of the sources, which are ω_1 and ω_2, respectively. Therefore $\mathbf{E}_1\times\mathbf{B}_2$ and $\mathbf{E}_2\times\mathbf{B}_1$ give zero net contribution in the long run. To see it, consider,

$$\begin{aligned} & \int_0^T \cos(\omega_1 t)\cos(\omega_2 t)dt \\ = \quad & \frac{1}{2}\int_0^T \left[\cos((\omega_1+\omega_2)t)+\cos((\omega_1-\omega_2)t)\right] dt \end{aligned}$$

$$= \quad \frac{\sin((\omega_1+\omega_2)T)}{2(\omega_1+\omega_2)} + \frac{\sin((\omega_1-\omega_2)T)}{2(\omega_1-\omega_2)}, \tag{1}$$

thus,

$$\left|\int_0^T \cos(\omega_1 t)\cos(\omega_2 t)dt\right| \;\leq\; \frac{1}{2|\omega_1+\omega_2|} \;+\; \frac{1}{2|\omega_1-\omega_2|},$$

e.g., bounded by a time constant. Therefore, the average contribution of the coupling terms to the emitted power during the $(0, T)$ interval is,

$$\langle \Delta P\rangle_T \;\propto\; \frac{\int_0^T \cos(\omega_1 t)\cos(\omega_2 t)dt}{T},$$

and it is going to vanish in the limit of large T.

Thus,

$$\langle \mathbf{S}(\mathbf{x})\rangle \;=\; \langle \mathbf{S}_1(\mathbf{x})\rangle + \langle \mathbf{S}_2(\mathbf{x})\rangle,$$

from two simple harmonic sources of *different* frequencies.

Question 3 (7 pt)

a. Prove

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) \;=\; \hat{a}+\alpha,$$

where

$$\hat{D}(\alpha) \;\equiv\; e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}},$$

is the operator that creates a coherent state $|\alpha\rangle$ out of vacuum.

Answer: Using the identity,

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} \;=\; e^{[\hat{A},\ ]}\hat{B} \;\equiv\; \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{1}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]]...,$$

and letting $\hat{A} = -(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, $\hat{B} = \hat{a}$, we have,

$$[\hat{A},\hat{B}] \;=\; [-(\alpha\hat{a}^\dagger - \alpha^*\hat{a}),\ \hat{a}] \;=\; \alpha.$$

Since $[\hat{A},\hat{B}]$ is just a constant, $[\hat{A},\hat{B}]$ commutes with any operator, and therefore all the higher-order expansion terms vanish, and we are left with,

$$e^{-(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}\hat{a}e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \;=\; \hat{a}+\alpha,$$

which is what we want to prove.

b. Since $\hat{D}(\alpha)$ is called the *displacement operator*, it is reasonable to expect that they satisfy the law of vector addition,

$$\hat{D}(\alpha)\hat{D}(\beta) \;=\; \lambda\hat{D}(\alpha+\beta).$$

Prove that this is indeed true.

Answer: Since,

$$\hat{D}(\alpha)\hat{D}(\beta) \;=\; e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}e^{\beta\hat{a}^\dagger-\beta^*\hat{a}},$$

using the Baker-Hausdorff theorem,

$$e^{\hat{A}}e^{\hat{B}} \;=\; e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]},$$

where we let,

$$\hat{A} \;=\; \alpha\hat{a}^\dagger-\alpha^*\hat{a}, \quad \hat{B} \;=\; \beta\hat{a}^\dagger-\beta^*\hat{a},$$

which is applicable here because the commutator,

$$[\alpha\hat{a}^\dagger-\alpha^*\hat{a},\ \beta\hat{a}^\dagger-\beta^*\hat{a}] \;=\; \alpha\beta^*-\alpha^*\beta,$$

is just a number and commutes with any operators, including $\hat{A}$ and $\hat{B}$ themselves, we have,

$$e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}e^{\beta\hat{a}^\dagger-\beta^*\hat{a}} \;=\; e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}+\beta\hat{a}^\dagger-\beta^*\hat{a}+\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)} \;=\; e^{(\alpha+\beta)\hat{a}^\dagger-(\alpha^*+\beta^*)\hat{a}}e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)},$$

or,

$$\hat{D}(\alpha)\hat{D}(\beta) \;=\; e^{\frac{1}{2}(\alpha\beta^*-\alpha^*\beta)}\hat{D}(\alpha+\beta).$$

Bonus Question (4 pt)

In classical field theory, the Poynting vector is defined as,

$$\mathbf{S}(\mathbf{x},t) \;\equiv\; \frac{c}{4\pi}\mathbf{E}(\mathbf{x},t)\times\mathbf{B}(\mathbf{x},t).$$

Please propose a corresponding operator $\hat{\mathbf{S}}(\mathbf{x},t)$ for quantum field theory, expressed in $\{\hat{a}_{\mathbf{k}\lambda}(t), \hat{a}^\dagger_{\mathbf{k}\lambda}(t)\}$, and explain why your choice should be unambiguous.

Answer: I propose,

$$
\begin{aligned}
\hat{\mathbf{S}}(\mathbf{x},t) \quad = \quad \frac{c}{4\pi} \quad & \sum_{\mathbf{k},\lambda} i\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\left[\hat{a}_{\mathbf{k}\lambda}(t)e^{i\mathbf{k}\cdot\mathbf{x}} - \hat{a}^{\dagger}_{\mathbf{k}\lambda}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\epsilon_{\mathbf{k}\lambda} \times \\
& \sum_{\mathbf{k}',\lambda'} i\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}'}}}\left[\hat{a}_{\mathbf{k}'\lambda'}(t)e^{i\mathbf{k}'\cdot\mathbf{x}} - \hat{a}^{\dagger}_{\mathbf{k}'\lambda'}(t)e^{-i\mathbf{k}'\cdot\mathbf{x}}\right](\mathbf{k}' \times \epsilon_{\mathbf{k}'\lambda'}). \qquad (2)
\end{aligned}
$$

This transition from classical field variable $\mathbf{S}(\mathbf{x},t)$ to quantum field operator $\hat{\mathbf{S}}(\mathbf{x},t)$ should be unambiguous because $\hat{a}_{\mathbf{k}\lambda}(t)$, $\hat{a}^{\dagger}_{\mathbf{k}\lambda}(t)$ appear in the same $\hat{a}_{\mathbf{k}\lambda}(t)e^{i\mathbf{k}\cdot\mathbf{x}} - \hat{a}^{\dagger}_{\mathbf{k}\lambda}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}$ term together in both $\hat{\mathbf{E}}(\mathbf{x},t)$ and $\hat{\mathbf{B}}(\mathbf{x},t)$, so in fact any $\hat{E}_i(\mathbf{x},t)$, $\hat{B}_j(\mathbf{x},t)$ commute where i,j are Cartesian indices, therefore the order of which appears first in the cross product doesn't matter.