22.51 Quiz III (90 min, Chen&Kotlarchyk book only)

Question 1 (7 pt)

Instead of a constant perturbation $\hat{V}$ to the Hamiltonian, suppose the perturbation is oscillatory:

$$\hat{V}(t) \quad = \quad \hat{V}_0 \cos(\omega_0 t).$$

Derive something similar to Fermi's Golden Rule No. 2 and explain carefully each step of your derivation.

Answer: Starting with perturbation expansion of the time-evolution operator,

$$\hat{U}_I(t) \quad = \quad 1 + \frac{1}{i\hbar}\int_0^t dt' \hat{V}_I(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \hat{V}_I(t') \int_0^{t'} dt'' \hat{V}_I(t'') + ...,$$

and taking the first order term of the transition amplitude $c_{nm}(t) \equiv \langle n|\hat{U}_I(t)|m\rangle$,

$$c_{nm}^{(1)}(t) \quad = \quad \frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{nm}t'} \langle n|\hat{V}(t')|m\rangle.$$

When $\hat{V}(t) = \hat{V}_0 \cos(\omega_0 t)$, there is,

$$\begin{aligned}
c_{nm}^{(1)}(t) \quad &= \quad \frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{nm}t'} V_{0nm} \cos(\omega_0 t') \\
&= \quad \frac{1}{i\hbar} V_{0nm} \int_0^t dt' e^{i\omega_{nm}t'} \frac{e^{i\omega_0 t'} + e^{-i\omega_0 t'}}{2} \\
&= \quad \frac{1}{i\hbar} V_{0nm} \int_0^t dt' \frac{e^{i(\omega_{nm}+\omega_0)t'} + e^{i(\omega_{nm}-i\omega_0)t'}}{2}. \qquad (1)
\end{aligned}$$

At this point, it can be projected that,

$$W_{nm} \quad = \quad \frac{\pi}{\hbar}|V_{0nm}|^2 \left[\delta(E_n - E_m - \hbar\omega_0) + \delta(E_n - E_m + \hbar\omega_0)\right].$$

Mathematically, it can be shown that $\sin(\alpha x)/\alpha x$ coupled to $\sin(\alpha(x-1))/\alpha(x-1)$ leads to nowhere if $\alpha \to \infty$, whereas $\sin(\alpha x)/\alpha x$ coupled to $\sin(\alpha x)/\alpha x$ leads to a δ-function, thus putting the above projection on a rigorous footing.

Question 2 (6 pt)

Explain why $\hat{\mathbf{p}}_i \cdot \hat{\mathbf{A}}_i$ is considered a good approximation to the first-order electron-EM field coupling operator $(\hat{\mathbf{p}}_i \cdot \hat{\mathbf{A}}_i + \hat{\mathbf{A}}_i \cdot \hat{\mathbf{p}}_i)/2$ when i is a *bound* electron and the radiation considered is in the visible light region. Do you think it is still so when the radiation is X-ray? gamma-ray?

Answer: The part in $\hat{\mathbf{A}}_i$ that does not commute with $\hat{\mathbf{p}}_i$ is the $e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_i}$ or the $e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}_i}$ factor. Consider,

$$e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_i}\hat{\mathbf{p}}_i e^{-i\mathbf{k}\cdot\hat{\mathbf{x}}_i} \quad = \quad \hat{\mathbf{p}}_i + i\mathbf{k}\cdot[\hat{\mathbf{x}}_i, \hat{\mathbf{p}}_i] \quad = \quad \hat{\mathbf{p}}_i + i\mathbf{k}\cdot(i\hbar\mathbf{I}) \quad = \quad \hat{\mathbf{p}}_i - \hbar\mathbf{k}.$$

Then,

$$e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_i}\hat{\mathbf{p}}_i \quad = \quad (\hat{\mathbf{p}}_i - \hbar\mathbf{k})e^{i\mathbf{k}\cdot\hat{\mathbf{x}}_i}.$$

We would like to argue that the $\hbar\mathbf{k}$ (the photon momentum) contribution is somehow much less than the $\hat{\mathbf{p}}_i$ (the electron momentum) contribution, so the non-commutative part of $\hat{\mathbf{p}}_i \cdot \hat{\mathbf{A}}_i$ is much smaller than $\hat{\mathbf{p}}_i \cdot \hat{\mathbf{A}}_i$ itself. By the Heisenberg relation,

$$\Delta p \cdot \Delta x \quad \sim \quad \hbar,$$

thus the magnitude of $\hat{\mathbf{p}}_i$ is on the order of $\hbar/a$ where a is the atom's size, which is the likely range of the bound electron and is usually on the order of a few Å. Thus, the relative importance of $\hbar\mathbf{k}$ with $\hat{\mathbf{p}}_i$ is again a comparison between the wavelength of the photon with the size of the atom, as in the electric dipole approximation later on.

For visible light, the wavelength is on the order of 4000-7000Å, therefore this approximation is valid. For X-rays, the wavelength is on the order of Angstroms, so the approximation is icky but may still work. For gamma-rays, this definitively cannot work.

Question 3 (7 pt)

Suppose isotope A has $b_{\mathrm{coh}}(\mathrm{A})$ and $b^2_{\mathrm{inc}}(\mathrm{A})$, isotope B has $b_{\mathrm{coh}}(\mathrm{B})$ and $b^2_{\mathrm{inc}}(\mathrm{B})$. Is the above information sufficient to calculate the b_{coh} and b^2_{inc} of a mixture composed of x-portion of isotope A and $1-x$ portion of isotope B? If so, give the answer. If not, please explain.

Answer: Same as the rule of obtaining the variance of two groups of numbers where the average values and variances of each group are known, it can be rigorously shown that,

$$b^2_{\mathrm{inc}} \quad = \quad \left[x b^2_{\mathrm{inc}}(\mathrm{A}) + (1-x) b^2_{\mathrm{inc}}(\mathrm{B})\right] + \left[x(1-x)(b_{\mathrm{coh}}(\mathrm{A}) - b_{\mathrm{coh}}(\mathrm{B}))^2\right],$$

where the first bracket is the simple mixture of respective variances, and the second bracket is the variance when each group has no dispersion.

Bonus Question (6 pt)

In classical EM theory, radiation is shown to carry pressure (see Example 4.1). Given the thermal distribution Einstein derived (using quantum perturbation theory, no less),

$$\langle n_{\mathbf{k}\lambda} \rangle = \frac{1}{e^{\frac{\hbar\omega_k}{k_{\mathrm{B}}T}} - 1},$$

construct a semi-classical argument (kinetic, not thermodynamical) to calculate the wall pressure that a blackbody cavity of temperature T has to sustain.

Answer: Take a differential area dS. For a photon mode of a given direction which happens to take angle θ with the surface normal, there is a differential volume (shaded region) whereby if the photon is in it, then it is going to impact the surface during interval dt.

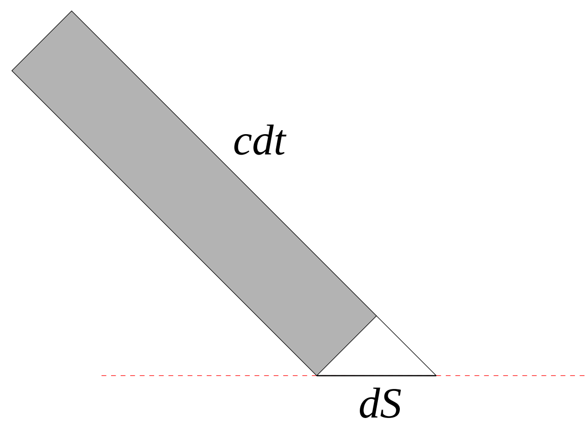

Figure 1: Cross section.

This differential volume is,

$$dV = cdtdS\cos\theta.$$

Since the photon could be anywhere is the blackbody volume V, the probability that this happens in dt is,

$$dP = \frac{dV}{V} = \frac{cdtdS\cos\theta}{V}.$$

Upon impact, the photon is going to be absorbed, leaving normal momentum $\hbar k\cos\theta$. On the other hand, in parallel, due to detailed balance this little area of wall lining is also responsible for replenishing the photon count of this particular mode equal to the rate of absorption by the emission process. The upshot is that the momentum transfer is $2\hbar k \cos\theta$ with the allowable solid $d\Omega$ spanning only half of the total solid angle ($4\pi/2 = 2\pi$). Thus, the pressure is,

$$p = \sum_{\mathbf{k}\in 2\pi,\lambda} \frac{2c\hbar k \cos^2\theta}{V} = \sum_{\mathbf{k}\in 4\pi,\lambda} \frac{\hbar\omega_k \cos^2\theta}{V}.$$

This happens to be the energy density expression with $\cos^2\theta$ thrown in. Since $\langle\cos^2\theta\rangle = 1/3$, there is,

$$p = \frac{u}{3},$$

where u is the photon energy per volume.