Notes on Adiabatic Invariants*

Ju Li

May 25, 1998

1 Introduction

Adiabatic invariance, an approximate conservation law, is a general result for any dynamical system that can be described by a Hamiltonian and which follows periodic motion. The problem was first considered by Einstein in 1911 as follows (see Fig. 1): suppose we have a pendulum that is hanging on the edge of a table with string length l_0. From elementary mechanics we know that it will oscillate periodically with frequency $\omega_0 = \sqrt{g/l_0}$, and the total energy E_0 is a constant of motion which determines the amplitude of the oscillation. Now, suppose one *slowly* pulls up the string, such that after a long time T, by long I mean T is much greater than the oscillation period of the pendulum, the string length becomes l_1. Apparently the oscillation frequency corresponding to the new string length has changed to $\omega_1 = \sqrt{g/l_1}$. The question is, what becomes of the oscillation amplitude, or total energy E_1? (suppose there is no dissipation.)

At first sight this problem seems ill-defined, because it does not specify how one should pull the string: should one pull it with constant velocity, constant velocity plus a sinusoidal part, or some increasing velocity? And indeed, the exact solution to the general problem would

*This is the topic of my oral presentation in MIT 8.614J *Introduction to Plasma Physics II*, taught by Prof. B. Coppi in Spring, 1998. Comments and suggestions are welcome to *liju99@mit.edu*.

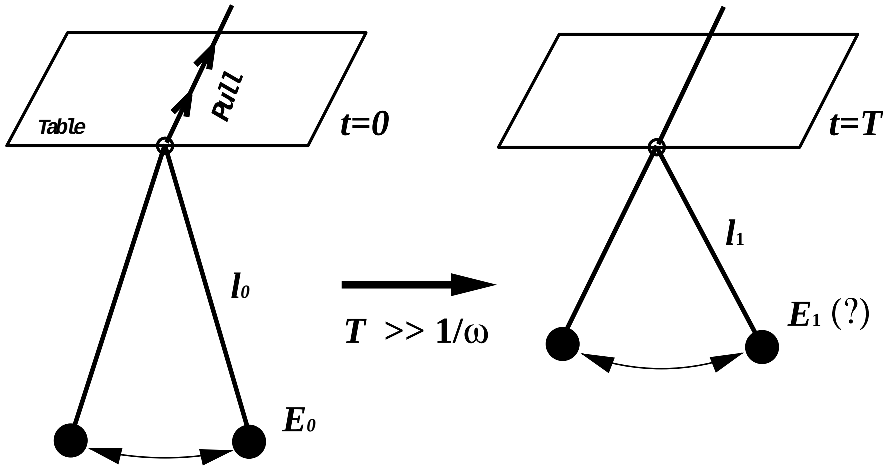

Figure 1: Pull up the pendulum gently.

require a computer, and the final result will depend on how the string is pulled. The point is that if one pulls it slowly *enough*, then no matter how it is pulled, in the end you'll find the following approximate relation to hold[1],

$$\frac{E_0}{\omega_0} = \frac{E_1}{\omega_1} + \text{small error}. \tag{1}$$

How small is the error? Well, what is the small parameter in this problem? Clearly it is $2\pi/\omega T$, the ratio of the oscillation period to the pulling interval. So could the error be as small as $2\pi/\omega T$, $(2\pi/\omega T)^2$, etc., if it is small? It turns out that the approximation is even better: the error made could be shown to be asymptotically smaller than any algebraic powers of $2\pi/\omega T$ (transcendentally small), something like $\exp(-\omega T/c)$, where c is a positive constant[1].

One can appreciate this result by relating to quantum mechanics. We know that the energy levels of a simple harmonic oscillator are $E = (n + 1/2)\hbar\omega$, a discrete instead of continuous spectrum. So E/ω in Eqn. (1) can be thought of as proportional to the quantum number n, which must be an integer and can only change by a finite amount. Since changing the system adiabatically should only induce continuous or infinitesimal changes to the properties, one could argue that n should not change, and thus E/ω in Eqn. (1) should stays constant. This

[1]This is for continuous string length function $l(t)$. It was established by Kulsrud[1] that if $l(t)$'s nth-order derivative is discontinuous, then the error $\sim (2\pi/\omega T)^{n+1}$.

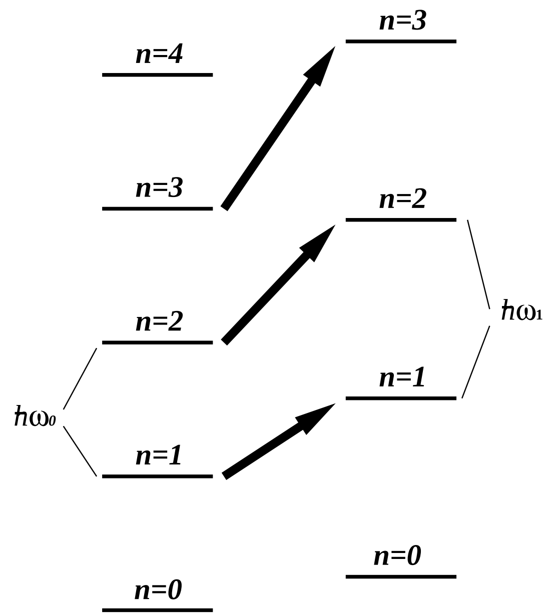

Figure 2: Adiabatic lifting: though the system has been altered, occupation of a specific state does not change if the alterations are made slowly.

unchanged occupation of an eigenstate despite changes to the system is called adiabatic lifting (Fig. 2), which can be proved rigorously in quantum mechanics[8, 9].

The above argument is physically illuminating. But a general proof[2, 3] can be given, entirely within the framework of classical mechanics, that for any dynamical system with equation of motion[2]

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \tag{2}$$

where $\mathcal{H} = \mathcal{H}(p, q|\lambda)$, λ are external parameters such as the string length l or the gravitational acceleration g in the above pendulum problem, and which follows periodic motion when λ is *fixed*, has an adiabatic invariant

$$A = \oint p dq, \tag{3}$$

that would remain *nearly* constant even if λ is changed adiabatically, by which we mean the timescale of changes in λ is much greater than the oscillation period. The error made would be transcendentally small.

[2]From now on I will omit the subscript i of a specific coordinate-momentum pair. The result is general for any multidimensional system.

Take the form of a simple harmonic oscillator which the pendulum problem can be reduced to. It has the familiar Hamiltonian

$$\mathcal{H}(p,q|m,\omega,..)=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2, \tag{4}$$

where we usually regard the mass m and the intrinsic frequency ω as constants, but here should be considered as external parameters (denoted genericly by λ) which are potentially subject to change. For instance, we can imagine a spring-and-ball model where the spring constant depends on, say, surrounding temperature. The question is then, what will be the oscillation amplitude in July if we set the spring to oscillate in January, if there is no dissipation?

The phase space trajectory of the simple harmonic oscillator is a closed ellipse (Fig. 3) when λ is fixed, with maximum displacement $\sqrt{2E/m\omega^2}$ and maximum momentum $\sqrt{2mE}$, so the area enclosed is $A=2\pi E/\omega$, which is our adiabatic invariant by definition. Now, if one slowly changes λ as time goes on, the trajectory would still be similar to ellipses but would fail to close exactly each time, and in general will spiral around and slowly deform, until a much later time, λ has changed from λ_0 to λ_1. If we stop changing λ then, the trajectory would still be an ellipse, but with a different shape because firstly λ has changed, and secondly the total energy E also changed from E_0 to E_1 because changing λ input energy to the system. The claim is that the area enclosed by the initial and final ellipses are approximately equal.

One may already observe its similarity with Liouville's theorem and hope that it might be a simple corollary of that theorem. However various attempts by the author have failed, mainly because it is an exact theorem for differential volume element $dpdq$, while the adiabatic theorem is *not* exact and deals with global properties, and somehow the condition of periodic motion for fixed λ has to be used. We know that any prescribed volume in phase space will quickly develop into "noodles" which has chaotic behavior, making the proof more difficult. The readers are encouraged to try this route and maybe, it can be shown after all.

In section 2, I list some of the theorem's applications to charged particle motion. A quick proof of the theorem, largely following that of Landau and Lifshitz[2], are given in section 3. In section 4, I will make some connections with quantum mechanics.

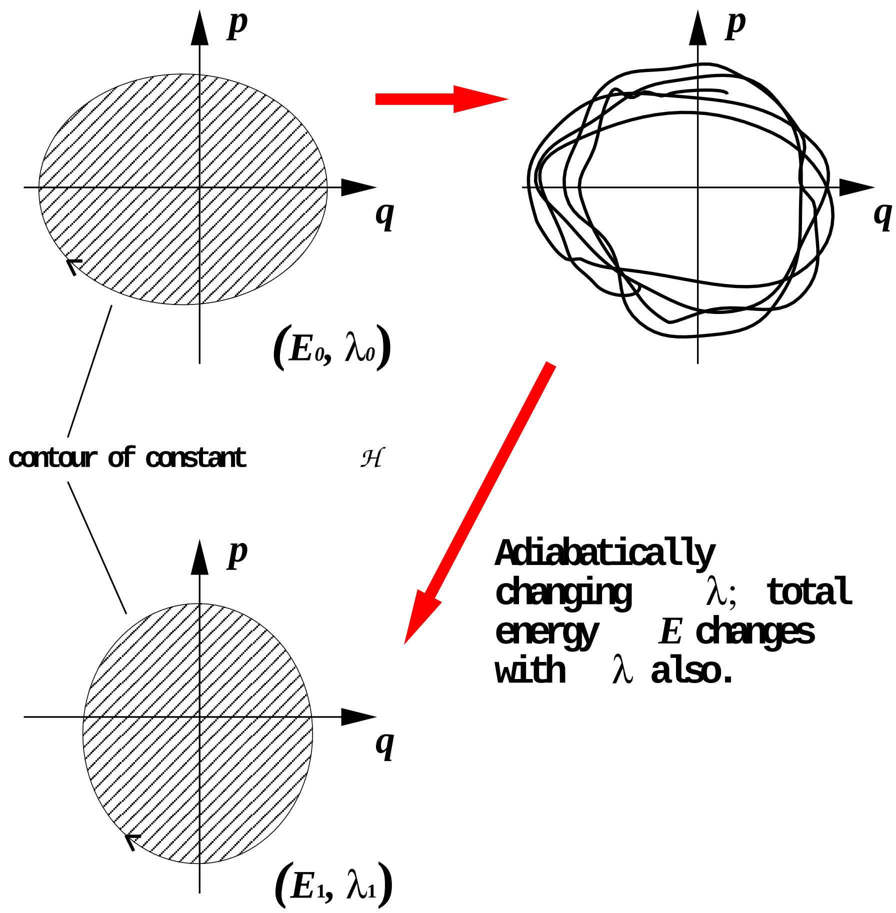

Figure 3: Phase space trajectories before, during and after changing λ adiabatically.

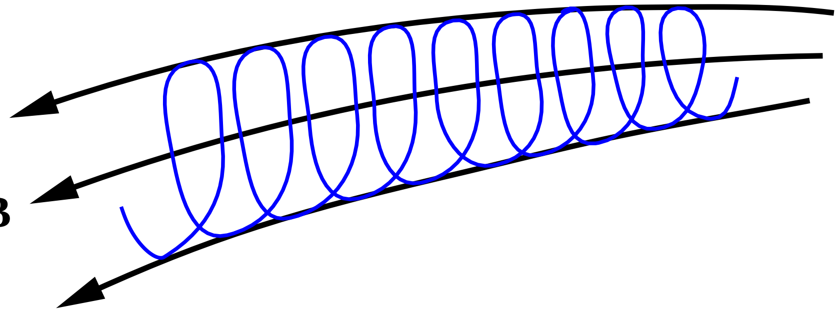

Figure 4: Charged particle gyrates as it "slides" along the magnetic field line.

2 Applications to Charged Particle Motion

There are three adiabatic invariants in a magnetic mirror confined plasma, each corresponding to a periodic motion with larger temporal and spatial scale than the previous ones.

The First Adiabatic Invariant is the magnetic moment μ, whose correspondent periodic motion is the Lamor gyration (Fig. 4). It is defined (in the non-relativistic limit) as

$$A_1 \propto \mu \equiv \frac{mv_\perp^2}{2B}, \tag{5}$$

which has the physical interpretation of diamagnetic current multiplied by the loop area[3], and is also proportional to the external magnetic flux through the loop. Its constancy is a very good approximation if

$$\left|\frac{B}{v_{\|}\partial B/\partial z}\right| \text{ and } \left|\frac{B}{\partial B/\partial t}\right| \quad \gg \quad \left|\frac{2\pi}{qB/m}\right|, \tag{6}$$

i.e., the timescale of changes in magnetic field that the particle feels is much greater than the particle Lamor gyration period.

This result can be shown from the Lagrangian of a charged particle in a magnetic field

$$\mathcal{L} = \frac{1}{2}m\dot{\mathbf{x}}^2 + q\dot{\mathbf{x}} \cdot \mathbf{A}, \tag{7}$$

where $\mathbf{A} = \mathbf{A}(\mathbf{x})$ is the magnetic vector potential. The relevant canonical coordinate in this

[3]For relativistic treatments see [4], pp. 588-593.

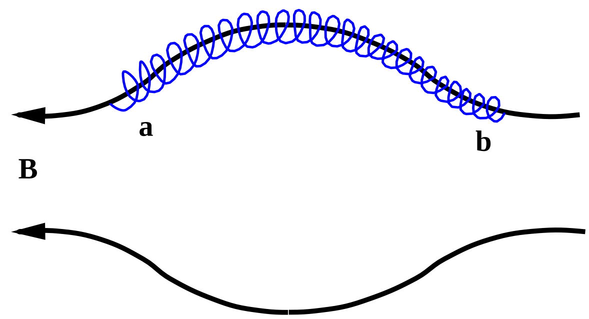

Figure 5: Charged particle bounces back and forth in a magnetic mirror configuration.

situation is the gyration angle θ, and so

$$\mathcal{L} = \frac{1}{2}mr^2\dot{\theta}^2 + qr\dot{\theta}A_\theta + \text{other terms}, \tag{8}$$

where the other terms are irrelevant because they can be regarded as external parameters that are held *fixed* right now. And so, the corresponding canonical momentum is

$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} + qrA_\theta, \tag{9}$$

and

$$\begin{aligned} A_1 &= \oint p_\theta d\theta \\ &= mr_L^2\dot{\theta} \cdot 2\pi + q\oint \mathbf{A} \cdot d\mathbf{l} \\ &= 2\pi m r_L^2 \dot{\theta} + qB \cdot \pi r_L^2, \end{aligned} \tag{10}$$

where we took the direction of $\mathbf{B}$ to be the positive sense of the integral. Since $\dot{\theta} = -qB/m$ in that sense, there is

$$\begin{aligned} A_1 &= -2\pi m v_\perp^2 \frac{m}{qB} + qB\pi v_\perp^2 \frac{m^2}{q^2B^2} \\ &= -\frac{2\pi m}{q} \cdot \frac{mv_\perp^2}{2B} \\ &\propto \mu. \end{aligned} \tag{11}$$

Adiabatic invariance of μ is broken when Eqn. (6) is violated, for example during magnetic pumping or cyclotron heating. For details see reference [5], pp. 44-45.

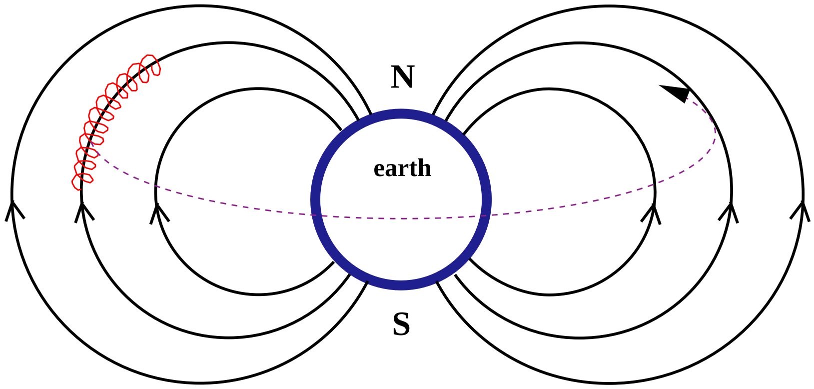

Figure 6: Charged particles slowly "precess" around the earth, in addition to the bouncing motion, due to the curvature of the field lines.

The Second Adiabatic Invariant, J, also called the *longitudinal invariant*, corresponds to following situation (Fig. 5): if μ is invariant, then $\frac{1}{2}mv_\perp^2$ increases with B. Yet, $\frac{1}{2}mv^2$ must be conserved in a static magnetic field; so, if we have bottle-shaped magnetic "mirrors"[4], such that the field at the bottleneck is strong enough, $v_{||}$ can be reduced to 0 at some *turning points* (a and b), and be reflected. If this is the case, and we have two opposite mirrors, then the bouncing back and forth between the two mirrors constitutes a periodic motion on its own right, which leads to an adiabatic invariant

$$A_2 = J \equiv \int_a^b mv_{||} ds, \tag{12}$$

where ds is a differential length on the path, which is the appropriate canonical coordinate. As with all adiabatic invariants, J is also "conserved" even if the magnetic field has a slow time dependence.

An application of the adiabatic invariance of J is to infer the pattern of macroscopic motion of charged particles trapped by earth's magnetic field in the *magnetosphere*[5] (Fig. 6), whose configuration is similar to the magnetic bottle in Fig. 5, which induces the particles to bounce back and forth between the poles. However because the magnetic field lines are curved (with radius of curvature $\mathbf{R}$), in addition to the bouncing motion there are so-called

[4]It can only look like that because $\nabla \cdot \mathbf{B} = 0$.

[5]The free electrons and ions in the magnetosphere come partly from the ionosphere below, and partly from the solar wind. The entrance of the fast electrons into earth's atmosphere near the north and south poles, due to the bouncing motion discussed above, is the cause of aurora, a greenish light produced by oxygen atom.

curvature and grad-B drifts

$$\mathbf{v}_{\text{drift}} = \mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}(v_{\|}^2 + \frac{v_{\perp}^2}{2})\frac{\mathbf{R}\times\mathbf{B}}{R^2B^2}, \tag{13}$$

which makes the particle slowly drift *across* the field lines and *precess* around the earth. Let us consider static fields first: if the magnetic field around the earth is axially symmetric (which it isn't), then clearly the line of bouncing motion (LCB) will come back to itself after one round of precession. However, what happens if the magnetic field is not axially symmetric, such as due to the solar wind? Will the LCB come back to itself after one round of precession? One can show that indeed it does, using the adiabatic invariance of J: because any magnetic field line can be uniquely specified by its azimuthal angle ϕ and distance to earth r. At any given azimuthal angle (as the LCB finishes one round of precession), there is only one degree of freedom, r, that could have changed. But if J is invariant we would have one constraint, and no field lines nearby could magically come up with the same J. And so the LCB must come back to the original field line after one round of precession.

If that is the case, then the precession *itself* is also a periodic motion, which leads to the Third Adiabatic Invariant

$$\Phi \propto A_3 \equiv \oint \overline{v_{\text{drift}}} \cdot r d\phi. \tag{14}$$

One may view the motion as an "enlarged version" of the Lamor gyration, and in fact Φ is defined to be the total magnetic flux through the barrel shaped envelope of motion, which should remain nearly constant even if the earth's magnetic configuration changes with time, so long as the timescale of changes is much greater than the precession period. That, however, is often not satisfied because the precession period of a charged particle is quite long. And so the invariance of A_3 is not as robust as A_1 or A_2.

In the above process of constructing $A_2(A_3)$, we see that one invokes the adiabatic invariance of $A_1(A_2)$, which has a smaller scale, and in fact is the basic constituent of the larger motion. This "bootstrapping" procedure of constructing a hierarchy of invariants is a beautiful result of physics.

3 Proof of the Adiabatic Theorem

Let me define the action integral[6] (adiabatic invariant) to be a *function*

$$A(E,\lambda) \equiv \oint pdq, \tag{15}$$

of E and λ which has the interpretation that if we freeze λ at the moment and let the system go at constant total energy E, A is the phase space area enclosed by the orbit during one period. To reiterate, A depends on λ because λ determines the Hamiltonian; A depends on E because it is the energy we let the system go (see Fig. 7); A does not depend on (p,q) because it does not matter where to start on a closed orbit.

Lemma I:

$$\left.\frac{\partial A}{\partial E}\right|_{\lambda} = \oint dt \equiv P(E,\lambda) \tag{16}$$

where P is the period of the orbit, a function of E and λ.

Proof:

We want to investigate how much A changes if one varies E: $E \to E + \delta E$, but fixing λ. Let me draw orbits $\mathcal{H}(p,q|\lambda) = E$ and $\mathcal{H}(p,q|\lambda) = E + \delta E$ on Fig. 7. By definition, δA is the area of the shaded layer,

$$\delta A = \oint dl \cdot \delta h, \tag{17}$$

where dl is a segment of the circular path[7]

$$dl = \sqrt{(dp)^2 + (dq)^2}, \tag{18}$$

and δh is the layer thickness. Since the orbits are nothing other than contour lines of the

[6] There seems to be some overlap of nomenclature here, because $A' = \int_{q(t_i)}^{q(t_f)} \mathcal{L}(q,\dot{q})dt$ is also called the *action* in the path integral formulation of classical and quantum physics, $\delta_q A' = 0$. They are closely related, however, since $\mathcal{L} = p\dot{q} - \mathcal{H}$, thus $A' = \int_{p(t_i),q(t_i)}^{p(t_f),q(t_f)} pdq - \mathcal{H}dt$, (from $\delta_{(p,q)} A' = 0$ we can derive Hamilton's equations), and so if we pick a closed trajectory, $A' = A - PE$ where E is the total energy and P is the period. As we shall see, this closely resembles the statistical mechanics formula $F = U - TS$.

[7] I am a little worried about the dimension of this quantity, but one can always first nondimensionalize p and q, and then proceed.

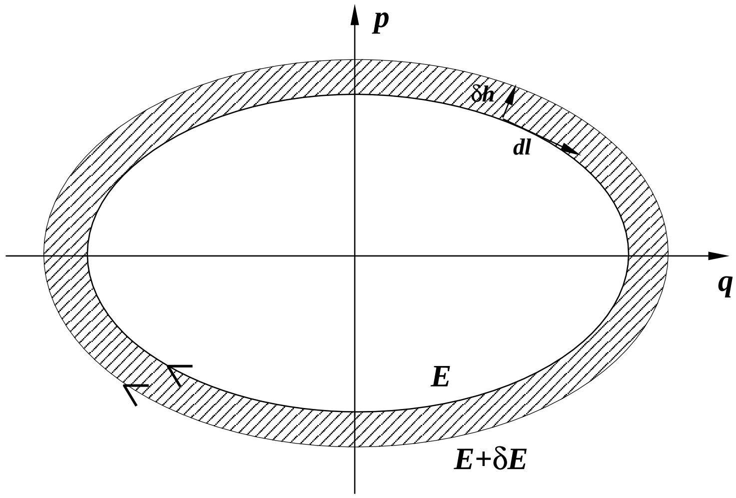

Figure 7: Two different orbits due to different E.

function $\mathcal{H}(p, q|\lambda)$, δh must be in the same direction as the gradient of $\mathcal{H}$ in (p, q) space,

$$\nabla\mathcal{H} = \left(\frac{\partial\mathcal{H}}{\partial p}, \frac{\partial\mathcal{H}}{\partial q}\right) = (\dot{q}, -\dot{p}) \tag{19}$$

with

$$|\nabla\mathcal{H}| = \sqrt{\dot{q}^2 + \dot{p}^2}, \tag{20}$$

and so the layer thickness must be

$$\delta h = \frac{\delta E}{|\nabla\mathcal{H}|} \tag{21}$$

at each point. Thus,

$$\begin{aligned} \delta A \quad &= \oint \sqrt{(dp)^2 + (dq)^2} \cdot \frac{\delta E}{\sqrt{\dot{q}^2 + \dot{p}^2}} \\ &= \delta E \oint dt. \end{aligned} \tag{22}$$

and the lemma is proved.

Eqn. (16) can be applied to simple harmonic oscillators. Since the period, $P = 2\pi/\omega$, is independent of E, and obviously $A(E = 0, \lambda) = 0$, there must be $A(E, \lambda) = 2\pi E/\omega$, in agreement with Eqn. (1).

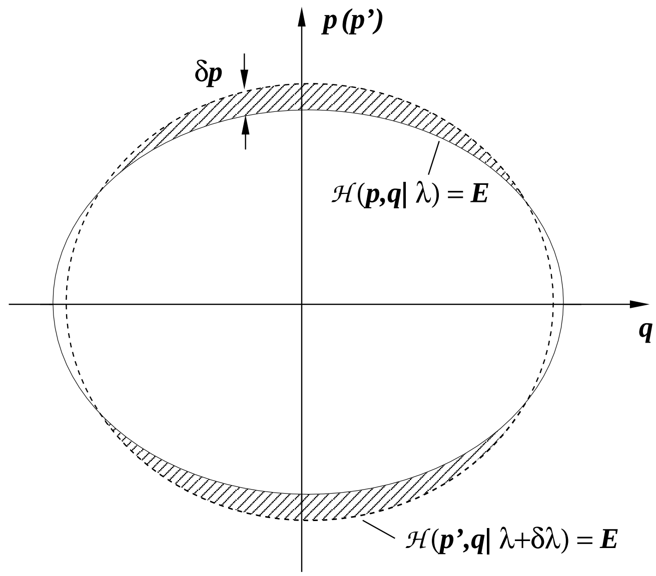

Figure 8: Two different orbits due to different λ.

Lemma II:

$$\left.\frac{\partial A}{\partial \lambda}\right|_E = -\oint dt \frac{\partial \mathcal{H}}{\partial \lambda} = -P\overline{\frac{\partial \mathcal{H}}{\partial \lambda}}. \tag{23}$$

Proof:

This corresponds to a slightly different picture (Fig. 8). The new orbit is now defined by

$$\mathcal{H}(p', q|\lambda + \delta\lambda) = E, \tag{24}$$

while the old one is defined by

$$\mathcal{H}(p, q|\lambda) = E, \tag{25}$$

where I choose the same q for the two, but p would differ slightly,

$$\delta p = p' - p. \tag{26}$$

To leading order, δA is the area of the shaded strip (those left out on the two sides are of higher order because the width of the orbit itself vanishes there). And so

$$\delta A = \oint dq \cdot \delta p. \tag{27}$$

What is δp? If we differentiate Eqn. (24) and compare with Eqn. (25), there is

$$\frac{\partial \mathcal{H}}{\partial p}\delta p + \frac{\partial \mathcal{H}}{\partial \lambda}\delta\lambda = 0, \tag{28}$$

and thus

$$\delta p = -\frac{\partial \mathcal{H}/\partial\lambda}{\partial \mathcal{H}/\partial p}\delta\lambda = -\frac{\partial \mathcal{H}/\partial\lambda}{\dot{q}}\delta\lambda. \tag{29}$$

So

$$\begin{aligned} \delta A &= \oint dq \left(-\frac{\partial \mathcal{H}/\partial\lambda}{\dot{q}}\right)\delta\lambda \\ &= -\delta\lambda \oint dt \frac{\partial \mathcal{H}}{\partial\lambda}, \end{aligned} \tag{30}$$

which I choose to write as

$$\delta A = -\delta\lambda \cdot P\overline{\frac{\partial \mathcal{H}}{\partial\lambda}}, \tag{31}$$

where $\overline{\partial\mathcal{H}/\partial\lambda}$ is defined to be the circular time average of $\partial\mathcal{H}/\partial\lambda$ as λ is frozen and (p, q) traces out a closed orbit. And so the lemma is proved.

With lemma I and II, we can prove the theorem. Notice that A has a significant dependence on both E and λ; fortunately, as a dynamical system evolves according to Eqn. (2), E changes with λ also, and the combined effect on A largely cancels out.

Since

$$\begin{aligned} \dot{E} = &\ \dot{\mathcal{H}} \\ = &\ \frac{\partial \mathcal{H}}{\partial p}\dot{p} + \frac{\partial \mathcal{H}}{\partial q}\dot{q} + \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda} \\ = &\ \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda}, \end{aligned} \tag{32}$$

as the first two terms exactly cancel[8]. So there is

$$\dot{A}(E,\lambda) = \left.\frac{\partial A}{\partial E}\right|_{\lambda}\dot{E} + \left.\frac{\partial A}{\partial \lambda}\right|_{E}\dot{\lambda}$$

[8] One notices its similarity with the Hellmann-Feynman theorem in quantum mechanics which, combined with the Born-Oppenheimer approximation, allows one to derive effective forces between atoms.

$$
\begin{aligned}
&= \quad P \cdot \frac{\partial \mathcal{H}}{\partial \lambda}\dot{\lambda} - P\overline{\frac{\partial \mathcal{H}}{\partial \lambda}} \cdot \dot{\lambda} \\
&= \quad P\dot{\lambda}\left(\frac{\partial \mathcal{H}}{\partial \lambda} - \overline{\frac{\partial \mathcal{H}}{\partial \lambda}}\right).
\end{aligned} \tag{33}
$$

Here comes the crucial insight: at any given moment, the growth rate of A, $\dot{A}$, does not necessarily have to be small. It could be algebraicly small, as $\dot{\lambda}$, but not exponentially small. However, the *time* average of $\dot{A}$ is going to be very small, since

$$
\langle \dot{A} \rangle = \langle P\dot{\lambda}\left(\frac{\partial \mathcal{H}}{\partial \lambda} - \overline{\frac{\partial \mathcal{H}}{\partial \lambda}}\right)\rangle, \tag{34}
$$

where the $\langle\rangle$ average is to be taken over many cycles; because $\partial \mathcal{H}/\partial \lambda$ must be a fast oscillatory quantity as (p,q) spirals around, with timescale $\sim P$, but both $\dot{\lambda}$ and P, which does not depend on (p,q), change slowly with timescale of the "pulling interval" T. So for

$$
\frac{\partial \mathcal{H}}{\partial \lambda} - \overline{\frac{\partial \mathcal{H}}{\partial \lambda}},
$$

$P\dot{\lambda}$ would just look like a constant and can be pulled out of the average. Thus,

$$
\langle \dot{A} \rangle \approx P\dot{\lambda}\langle\frac{\partial \mathcal{H}}{\partial \lambda} - \overline{\frac{\partial \mathcal{H}}{\partial \lambda}}\rangle \approx 0, \tag{35}
$$

and the adiabatic theorem is proved. One can already sense that whatever small terms the approximation (35) leaves behind should not be any algebraic powers of P/T.

⇒ Exactly how small is ΔA?

Let the adiabatic changes to λ happen at $t = -\infty$ and stop at $t = +\infty$. Then

$$
\Delta A = \int_{-\infty}^{+\infty} \dot{A} dt = \int_{-\infty}^{+\infty} P\dot{\lambda}\left(\frac{\partial \mathcal{H}}{\partial \lambda} - \overline{\frac{\partial \mathcal{H}}{\partial \lambda}}\right) dt. \tag{36}
$$

Define canonical time (or angular) variable

$$
\tau(q|E,\lambda) = \int_{q_{\min}}^{q} \frac{dq}{\dot{q}}, \tag{37}
$$

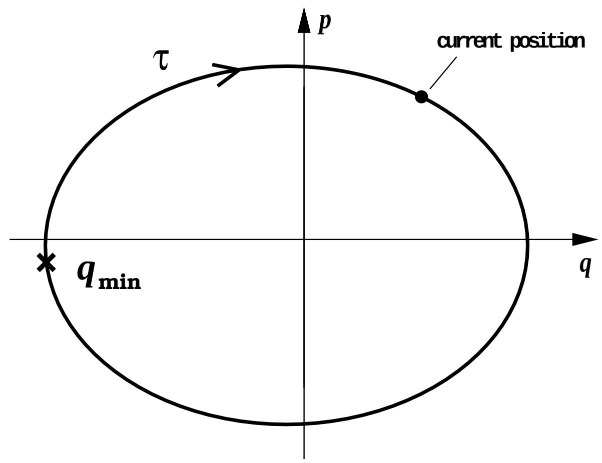

Figure 9: Definition of the canonical time τ.

as shown in Fig. 9, which is just the time it takes for the trajectory to go from q minimum to the current position on the closed orbit that it belongs to, if λ is fixed. Clearly $\partial\mathcal{H}/\partial\lambda$ is a periodic function of τ with period P, and so could be expanded into a Fourier series

$$\frac{\partial\mathcal{H}}{\partial\lambda} = \overline{\frac{\partial\mathcal{H}}{\partial\lambda}} + \sum_{n=1}^{\infty}\left\{a_n(E,\lambda)\exp(i\frac{2\pi n}{P}\tau) + c.c.\right\}, \tag{38}$$

where we have explicitly taken out the $n = 0$ term which is nothing other than $\overline{\partial\mathcal{H}/\partial\lambda}$. And following Eqn. (36), there is

$$\Delta A = \sum_{n=1}^{\infty}\left\{\int_{-\infty}^{+\infty} P(E,\lambda)\dot{\lambda}a_n(E,\lambda)\frac{dt}{d\tau}\exp(i\frac{2\pi n\tau}{P})d\tau + c.c.\right\}. \tag{39}$$

If $\dot{\lambda} = 0$, then t and τ are in phase and $dt/d\tau \equiv 1$. If λ varies, but slowly, then one still has $dt/d\tau \approx 1$; but more importantly, $dt/d\tau - 1$ has no high frequency components because the phase between t and τ are, by all means, nearly synchronous. Thus in the integrand of Eqn. (39), $\exp(i2\pi n\tau/P)$ is the fast oscillatory component, but $P(E,\lambda)$, $\dot{\lambda}$, $a_n(E,\lambda)$ and $dt/d\tau$ are all slow variables of τ, providing only the envelope as illustrated in Fig. 10. And the total integral will be small indeed, due to the myriad of cancellations.

Asymptotic evaluation of integrals have three main contributors: endpoints, singularities and saddle points, the latter two involving complex plane structures[7]. Here because the integration is carried out from $-\infty$ to $+\infty$, there are no endpoint contributions. Saddle points

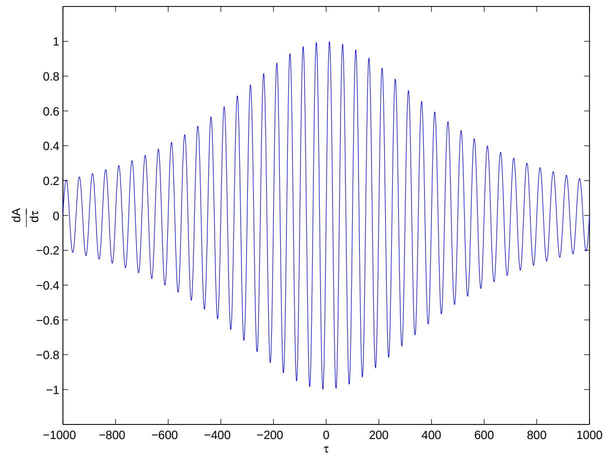

Figure 10: Illustration of $\dot{A}(\tau)$, integrand of Eqn. (39).

cannot exist because $P\dot{\lambda}a_n dt/d\tau$ has no matches for the fast oscillation of $\exp(i2\pi n\tau/P)$. The only possible contributors left are therefore singularities of $P\dot{\lambda}a_n dt/d\tau$ on the complex τ-plane, and since it is a well-behaved function on the real axis with characteristic timescale T, one could argue that the singularities must be at least $\mathcal{O}(T)$ away from the real axis[9]. And so if we lift the integration contour from $(-\infty \to +\infty)$ to $(-\infty + ia \to +\infty + ia)$ on the upper half plane, the first singularity it encounters (for $n = 1$) will give the leading contribution. In this case, therefore,

$$\Delta A \sim \exp(-\text{const} \cdot \frac{T}{P}), \tag{40}$$

a transcendentally small quantity for small P/T.

All the ideas above can be illustrated by a fictitious example. Try

$$\int_{-\infty}^{+\infty} \frac{T^2}{\tau^2 + T^2} \exp(i\frac{2\pi\tau}{P}) d\tau, \tag{41}$$

and see how small it is.

If $\lambda(t)$ has a discontinuity at the nth-order derivative, then it is equivalent to introducing endpoints to the integration (39), which in general dominates over the singularity contributions. Here one can tilt the integration path 90° upward at the endpoint, and show that the

[9] An example is $1/(x^2 + T^2)$. A counter-example is $|x|$, which can be thought of as $\lim_{\epsilon\to 0}\sqrt{x^2+\epsilon^2}$, with branch points $\pm i\epsilon$ pinching down.

leading order contribution must scale as

$$\Delta A \sim \left(\frac{P}{T}\right)^{n+1}, \tag{42}$$

where the T^{-n-1} factor comes from dimensional arguments. Try

$$\int_0^{+\infty} \tau^n \exp(-\frac{2\pi\tau}{P}) d\tau \tag{43}$$

to get the basic idea.

4 Remarks

1. The adiabatic invariant $A(E, \lambda)$ is in fact the crucial connection between classical and quantum physics. The following formula[10]

$$A \equiv \oint pdq = 2\pi(n + \frac{1}{2})\hbar \tag{44}$$

is the famous Bohr-Sommerfeld quantization condition, which could be derived from the bound-state condition in the WKB approximation[8, 9] (so-called "small h" expansion), within quantum mechanics. The fundamental "density of state" it leads to,

$$dN = \frac{dpdq}{h}, \tag{45}$$

is used ubiquitously in statistical mechanics and plays a crucial role in "getting the numbers right", from blackbody radiation to the mass of a neutron star.

2. The Born-Oppenheimer approximation[11], crucial in the discussion of atomic interactions in condensed matter physics, interestingly is also called the "adiabatic approximation". The gist is that since the "fast" electronic degrees of freedom are so much faster than the "slow" ionic degrees of freedom, their relaxation looks almost instantaneous to the ions; or from the electronic points of view, the ions move so slowly that their coordinates are really the adiabatic variable λ, which could not alter the

[10] The factor 1/2 should be omitted in the hydrogen atom model.

quantum numbers of occupied electronic states. The Born-Oppenheimer approximation fails whenever the available energy is so high that the ionic motions couple to electronic excitations.

3. Adiabatic lifting is a useful tool in regularizing quantum scattering problems and field theory[12], since it provides a well defined initial and final state for the system. The multitude of tricks involving small ϵ in the complex $(\mathbf{k}, \omega)$ plane after Fourier transform (whose rigorous definition *requires* square-integrability), can all be interpreted by adiabatic lifting with time-dependence $e^{-\epsilon|t|}$.

4. There are some symbolic correspondences between Eqn. (16) and the statistical mechanical formula[10]

$$\left.\frac{\partial S}{\partial E}\right|_{N,V} = \frac{1}{T}, \tag{46}$$

as it is well-known that there are a lot of such correspondences between time and inverse temperature. One cannot help but notice that

$$A' = A - PE, \tag{47}$$

introduced in the footnote of section 3, looks very similar to

$$F = U - TS \tag{48}$$

in statistical mechanics where F is the Helmholtz free energy. The reason might be that both formulations (Lagrangian to Hamiltonian dynamics, micro-canonical to canonical ensemble) involve Legendre transforms.

5. Lastly, in some quantum oscillator problems, such as a spin under external magnetic field, not only the oscillation amplitude, but also the phase of the oscillation, has a beautiful result if the relevant external parameters (such as the magnetic field) are varied adiabatically. The extra phase depends only on geometry, and is called the "Berry's phase"[9]. I am not sure whether similar examples exist in classical mechanics, but it might be not as important because phase only matters greatly to wave functions, which could interfere.

References

[1] R. M. Kulsrud, Phys. Rev. **106**, 205 (1957).

[2] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd rev. English ed., Pergamon Press, New York (1988).

[3] M. Kruskal, Conference Proceedings of Plasma Physics and Controlled Nuclear Fusion Research, Salzburg, 4-9 Sept. 1961.

[4] J. D. Jackson, *Classical Electrodynamics*, 2nd ed., Wiley, New York (1975).

[5] F. F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, Plenum Press, New York (1984).

[6] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill, New York (1973).

[7] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill, New York (1978).

[8] A. Messiah, *Quantum Mechanics*, Wiley, New York (1976).

[9] J. J. Sakurai, *Modern Quantum Mechanics*, Addison-Wesley, Reading, Mass. (1994).

[10] K. Huang, *Statistical Mechanics*, Wiley, New York (1987).

[11] M. Born and K. Huang, *Dynamical Theory of Crystal Lattices*, Clarendon Press, Oxford (1954).

[12] R. D. Mattuck, *A Guide to Feynman Diagrams in the Many-body Problem*, McGraw-Hill, New York (1976).